A 'Binary' System for Complex Numbers

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Unclassified

A number system with the complex number $-1+i$ as base is developed. This permits the representation in binary form of any complex number $a+bi$ with a and b integral or of the form $k/2^n$. In the latter case a separatrix is used to indicate negative powers of the base.

Computer operations with complex numbers are usually performed by dealing with the real and imaginary parts separately and combining the two as a final operation. It might be an advantage in some problems to treat a complex number as a unit and to carry out all operations in this form.

The number system to be described permits the representation of a complex number as a single binary number to a degree of accuracy limited only by the capacity of the computer. It is binary in that only the two symbols 1 and 0 are used; however, the base is not 2, but the complex number $-1+i$. The quantity $-1-i$ would be equally suitable, and, in fact, for real numbers it is immaterial which of these two we consider the base.

The first few powers of $-1+i$ are

1	$-1+i$	5	$4-4i$
2	$-2i$	6	$8i$
3	$2+2i$	7	$-8-8i$
4	-4	8	16

We have, for example, the following equivalents:

−4	1 0 0 0 0	0	0
−3	1 0 0 0 1	1	1
−2	1 1 1 0 0	2	1 1 0 0
−1	1 1 1 0 1	3	1 1 0 1

All the arithmetical operations can be performed on these numbers if the proper rules are observed. Corresponding to the ordinary (computer) rules, 1 + 1 = 1 0 and 1 + 1 1 1 . . . (to limit of machine) = 0, we have the rules 1 + 1 = 1 1 0 0 and 1 1 + 1 1 1 = 0. For example, to add 1 1 1 0 1 0 0 0 1 (= 5) and 1 0 0 0 1 0 0 0 1 (= 13),

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  1 1 1 0 1 0 0 0 1
+ 1 0 0 0 1 0 0 0 1
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  1 0 0 0 0 1 1 0 0 (= 18)
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The two 1's in the rightmost position become 1 1 0 0. The two 1's representing $(-1+i)^4$ become 1 1 0 0, and this combines with the initial 1's to produce $\begin{matrix}1\ 1\ 1\\1\ 1\ 1\end{matrix}$. Since $\begin{matrix}1\ 1\ 1\\1\ 1\end{matrix} = 0$, we have 1 0 0 0 0 1 1 0 0 as the final result.

Every integer, positive, negative or zero can be represented uniquely in the form $a_0 + a_1(-4) + a_2(-4)^2 + \ldots + a_k(-4)^k$ where each a is 0, 1, 2 or 3. To represent an integer to base $-1+i$, write it in powers of -4; the required representation will then be $a_k\, a_{k-1} \ldots a_1\, a_0$, where the digits are 0000, 0001, 1100 or 1101 according as a is 0, 1, 2 or 3 respectively.

For example,

$$46 = 3(-4)^2 + 1(-4) + 2.$$

Therefore,

$$46 = 312_{(-4)} = 1\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0.$$

Initial 0's are neglected; for example,

$$19 = 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 1 = 1\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 1.$$

The first few imaginary integers are

$-4i$	1 1 0 0 0 0	0	0
$-3i$	1 1 0 0 1 1	i	1 1
$-2i$	1 0 0	$2i$	1 1 1 0 1 0 0
$-i$	1 1 1	$3i$	1 1 1 0 1 1 1

Every imaginary integer can be represented uniquely in the form $a_0 + b_0(8i) + a_1(8i)^2 + b_1(8i)^3 + \ldots$ where $-6i \leq a \leq i$ and $-4 \leq b \leq 3$. To represent an imaginary integer to base $-1+i$, write it in powers of $8i$ with appropriate a's and b's; the required representation will then be this number with the digits replaced by their binary equivalents. If the binary equivalent for any digit (except the first) contains fewer than six bits, 0's are prefixed to round it out to six. For example, $77i = (-1)\cdot(8i)^2\, i + 2(8i) + (-3)\, i$. Therefore $77i$ = 1 1 1 0 0 1 1 0 0 1 1 0 0 1 1. Here 1 1 1 represents $-i$, 0 0 1 1 0 0 represents 2 and 1 1 0 0 1 1 represents $-3i$.

Any complex number $a + bi$ with a and b integral can be expressed as the sum of the real and imaginary parts. For example, -2 = 1 1 1 0 0 and $3i$ = 1 1 1 0 1 1 1, so that $-2 + 3i$ = 1 1 1 0 0 + 1 1 1 0 1 1 1 = 1 1 0 1 1 . There is thus a one-to-one correspondence between the binary numbers of this system and the complex numbers $a + bi$ with a and b positive or negative integers or zero.

By the use of 'decimals,' that is, negative powers of $-1+i$, it is possible to extend this system to the representation of all complex numbers of the form $a/2^m + (b/2^n)\,i$. For example, we have the following equivalents:

$-\frac{3}{4}$	. 1 1 0 1	$-\frac{3}{4}i$	1 1 1 . 0 1 1 1
$-\frac{1}{2}$	. 1 1	$-\frac{1}{2}i$	1 1 1 . 0 1
$-\frac{1}{4}$	. 0 0 0 1	$-\frac{1}{4}i$	. 0 0 1 1
$\frac{1}{4}$	1 . 1 1 0 1	$\frac{1}{4}i$	. 0 1 1 1
$\frac{1}{2}$	1 . 1 1	$\frac{1}{2}i$	. 0 1
$\frac{3}{4}$	1 . 0 0 0 1	$\frac{3}{4}i$	1 1 . 0 0 1 1

As with integers, the real and imaginary parts combine to form a single number. $\frac{1}{2} - \frac{3}{4}i$, for example, $= 1.11 + 111.0111 = 1.1011$.

Since a real number can be approximated to any desired degree of accuracy by a fraction of the form $a/2^n$, this number system will permit the representation of all complex numbers to a degree of accuracy limited only by the capacity of the computer.