Title: Dinner at the Rabbit Café: A Predator-Prey Investigation

Brief Overview:

The students will investigate how populations grow with and without another predator population. They will build possible models to show how two populations (one a predator and the other a prey) interact with each other. They will look at the trends both numerically and graphically. They will investigate the sensitivity of the parameters that affect the populations as time passes.

NCTM Content Standard/National Science Education Standard:

- Content Standards

    * Number and Operations
        * Understand meanings of operations and how they relate to one another
        * Compute fluently and make reasonable estimates
    * Algebra
        * Understand patterns, relations and functions
        * Represent and analyze mathematical situations and structures using algebraic symbols
        * Use mathematical models to represent and understand quantitative relationships
        * Analyze change in various contexts
    * Measurement
        * Understand measurable attributes of objects and the units, systems and processes of measurement
    * Data Analysis and Probability
        * Formulate questions that can be answered with data and collect, organize and display relevant data
    * Problem Solving
        * Build new mathematical knowledge through problem solving
        * Solve problems that arise in mathematics and in other contexts
        * Apply and adapt a variety of appropriate strategies to solve problems
        * Monitor and reflect on the process of mathematical problem solving
    * Reasoning and Proof
        * Make and investigate mathematical conjectures
    * Communication
        * Organize and consolidate their mathematical thinking through communication
        * Communicate their mathematical thinking coherently and clearly to peers, teachers and others

* Analyze and evaluate the mathematical thinking and strategies of others
* Use the language mathematics to express mathematical ideas precisely

* Connections
  * recognize and use connections among mathematical ideas
  * Understand how mathematical ideas interconnect and build on one another to produce a coherent whole
  * Recognize and apply mathematics in contents outside of mathematics

*Representation
  * Create and use representations to organize, record, and communicate mathematical ideas
  * Select, apply and translate among mathematical representations to solve problems
  * Use representations to model interpret physical, social and mathematical phenomenon

Virginia Standards of Learning (SOL)

Appropriate standards for this lesson cross-disciplinary lines and incorporate math, science and technology. A few of the relevant standards are listed below. Others may also be applicable.

COM Standards are also applicable to the TI 83 calculator that is used throughout Dinner At The Rabbit Café (computer programming standards)

COM.1 The student will apply programming techniques and skills to solve practical problems in mathematics arising from consumer, business, other applications in mathematics. Problems will include opportunities for students to analyze data in charts, graphs, and tables and to use their knowledge of equations, formulas, and functions to solve these problems.

COM.12 The student will translate a mathematical expression into a computer statement, which involves writing assignment statements and using the order of operations.

COM.13 The student will select and implement built-in (library) functions in processing data.

Probability and Statistics

*PS.1 The student will analyze graphical displays of data, including dotplots, stemplots, and histograms, to identify and describe patterns and departures from patterns, using central tendency, spread, clusters, gaps, and outliers. Appropriate technology will be used to create graphical displays.

Math Analysis

MA.9 The student will investigate and identify the characteristics of exponential and logarithmic functions in order to graph these functions and solve equations and practical problems. This will include the role of e, natural and common logarithms, laws of exponents and logarithms, and the solution of logarithmic and exponential equations. Graphing utilities will be used to investigate and verify the graphs and solutions.

Calculus

APC.5 The student will investigate derivatives presented in graphic, numerical, and analytic contexts and the relationship between continuity and differentiability. The derivative will be defined as the limit of the difference quotient and interpreted as an instantaneous rate of change.

APC.8 The student will apply the derivative to solve problems. This will include:

a) analysis of curves and the ideas of concavity and monotonicity;
b) optimization involving global and local extrema;
c) modeling of rates of change and related rates;
d) use of implicit differentiation to find the derivative of an inverse function;
e) interpretation of the derivative as a rate of change in applied contexts, including velocity, speed, and acceleration

* AP Calculus BC will also apply the derivative to solve problems. This will include

a) analysis of planar curves given in parametric form, polar form, and vector form, including velocity and acceleration vectors;
b) numerical solution of differential equations, using Euler's method;
c) geometric interpretation of differential equations via slope fields and the relationship between slope fields and the solution curves for the differential equations.

Biology

BIO.8 The student will investigate and understand how populations change through time. Key concepts include

a) evidence found in fossil records;
b) how genetic variation, reproductive strategies, and environmental pressures impact the survival of populations;
c) how natural selection leads to adaptations;
d) emergence of new species; and
e) scientific explanations for biological evolution.

BIO.9 The student will investigate and understand dynamic equilibria within populations, communities, and ecosystems. Key concepts include

a) interactions within and among populations including carrying capacities, limiting factors, and growth curves;
b) nutrient cycling with energy flow through ecosystems;
c) succession patterns in ecosystems;
d) the effects of natural events and human activities on ecosystems; and
e) analysis of the flora, fauna, and microorganisms of Virginia ecosystems including the Chesapeake Bay and its tributaries.

Computer/Technology Standards

In grades nine through twelve, technology continues to be integrated across the curriculum. The goal is that students in these grades achieve a higher level of mastery in the application of technology in their learning.

C/T12.2 The student will use application software to accomplish a variety of learning tasks.

- Use advanced features of word processing, desktop publishing, graphics programs, and utilities in learning activities.
- Use spreadsheets for analyzing, organizing and displaying numeric data graphically.

Grade/Level:

10-12 advanced algebra or precalculus classes

Duration/Length:

Approximately three 60 minutes classes

Student Outcomes:

Students will:

- Develop models to simulate population growth.
- Create data in Excel and TI-83 lists (sequence mode).
- Investigate how the parameters affect the final data.
- Learn to use recursively defined functions.
- Interpret different graphical representations of the data.

Materials and Resources:

- A computer with Excel
- TI-83 calculator
- Teacher notes and answer keys
- Student worksheets

Development/Procedures:

Lesson 1 Introduction to Population Growth: One Species Alone and the Beginning of a predator-Prey Relationship

Preassessment – Ask students for their ideas about how a population of one species of animals grows.

Launch – Students will work through Worksheet One to investigate when populations grow and when they drop.

Teacher Facilitation – Discuss how there can be two factors on a population change: a tendency to grow and a force that will decrease a population. This can be applied to a population of a predator-prey relationship. Lead students to adapt their model to accommodate both of these forces simultaneously. Students will be helped to understand that this is a vastly simplified version of all of the actual factors on a population, but it is beginning to simulate actual conditions. Students will be taught how to use the formulas in an Excel spreadsheet to model this recursive growth.

Student Application – Students will work collaboratively to develop the model and decide on some values of the parameters for the growth and decay of the populations of the predators and prey. They will learn to enter the information into an Excel spreadsheet to simulate the population growth. Their goal is to find values of the parameters that will lead to populations of both the predator and prey that will continue to thrive together.

Embedded Assessment - Listen to the student pairs / groups as they investigate. Students will complete the Excel worksheet, and they will write some analysis of their results (whether they felt they were successful or not).

Reteaching/Extension –

- Discussion of why one of the populations goes to zero (if that happens), and review how to work with the individual forces to make populations increase or decrease.
- Suggest other factors that could have been added to make the model more realistic.

Lesson 2 Refinement of the Model and the SEQUENCE mode on the TI-83

Preassessment – Discuss how the population models worked or did not work in the previous lesson. Ask students to write in specific ways how the model was appropriate.

Launch - Students will begin to work on Worksheet Two. They will read about recursively defined functions, and they will use the new mode on their TI-83's.

Teacher Facilitation - Explain what a recursively defined function is in mathematics. Discuss the SEQUENCE mode to simulate recursively defined functions.

Student Application – Students will use one of their models from the previous lesson in the SEQUENCE mode in their calculators to be sure that they can produce identical results. Students will then produce graphs of their populations and interpret what they see graphically.

Embedded Assessment – Student groups will discuss with the class their results, so that they can see what others accomplished, where some groups could not manage to keep both populations alive, or how different successful results occurred. They will answer questions about what their graphs tell them.

Reteaching/Extension

Lesson 3 Independent Student Work on an Extension of the Predator-Prey Model

Preassessment – Students will make sure that they can produce a set of parameter values so that the populations remain stable and both are viable.

Launch – Students will work without teach intervention on Worksheet Three.

Teacher Facilitation - Very little.

Student Application - Students use what they have learned to implement the modification of the model in pairs / groups

Embedded Assessment – The values of the parameters will be found, and interpretive answers will be written by the pairs/ groups.

Reteaching/Extension

Summative Assessment:

Worksheet Four (and Worksheet Three in Lesson 3)

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Dinner at the Rabbit Café

WORKSHEET ONE

Predator – Prey Relationships and the Significance of Predator Species

Predators are an important and integral part of any ecological community. They are one of the most significant factors in controlling natural populations. Predators keep the numbers of prey populations from varying too greatly. A given habitat can support a finite number of inhabitants. Nature has a number of means for controlling populations in any given ecosystem. Predators are one of the most important natural factors. There are many other factors, such as forest fires, floods, disease, famine, reproductive age limitations, drought and other natural disasters. And of course, we cannot overlook the impact of humans on the natural environment.

Predators are the best wildlife managers[1]. Animals that do not have predator species often pose the greatest challenges to wild life management specialists in reserves. Predators act as natural selection tools when they weed out the sick and old among a population. These are "easy prey" to feed upon. The healthy and vigorous animals are left to survive and reproduce when the weak and sick are removed. Just to show you that the numbers are significant, consider how much prey these predators eat: If you are a lion you need 20 to 30 animals a year. Of course, these are big animals like zebras and gazelles. Doesn't sound like much? In a given ecological unit (a habitat of defined or limited area), lions account for the taking of 40,000 to 72,000 animals per year! A solitary cheetah kills 150 animals per year, and a cheetah with cubs accounts for at least double that number.

Predation does not work alone as a limiting factor. Indeed, it is generally related to other factors in the environment, e.g. predation has an interesting relationship to disease and malnutrition. Research shows that when the intensity of predation on a given species increases, the occurrence of disease decreases.[2] (Although for our purposes we will simplify this scenario by using specific and limited factors that impact our prey population). Some people, like hunters, use the argument that if they were not allowed to kill deer for instance, then the deer would just overpopulate and die of malnutrition or starvation. We will leave that discussion for the biology teacher! The most important influence of predation is that it limits the prey population and keeps it below the carrying capacity of the prey's environment. Note: carrying capacity is defined as the number of individuals a given habitat can support.

For the purposes of this exercise, the situation has been simplified. In the initial model only the population of the prey species is considered. Note: In the absence of other

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[1] Page 406, The Serengeti Lion, A Study of Predator-Prey Relations, George B. Schaller, The University of Chicago Press, 1972.
[2] Schaller, page 202.

factors, populations grow at an exponential rate. The second model will take into account the effect that the presence of the predator species has on the prey population. The third model will bring in the "encounter" factor that is the probability that the predator will have the opportunity to feed on the prey. This is a simplified model based on the traditional Lotka-Volterra model, (more later on Lotka-Volterra) but one that lends itself for use with novice math modelers. For those who want to extend the model, consider adding birth and death rates, the distribution of males and females in the prey population and the reproductive age limits for the prey population. There are, of course, many other factors to consider. The Internet is a remarkable resource for determining what specific factors impact a certain population.

If you find the subject of predator-prey relationships and mathematical modeling interesting, you might want to do some research into the Lotka-Volterra Model for Populations. There is extensive material available on the topic. The model includes methods for handling most factors that have impact on population dynamics.

The Lotka-Volterra predator prey model equations are the basis of many models in predicting populations. Lotka's and Volterra's discoveries were independent. Volterra was studying the population dynamics of fish in the Italian fish markets and Lotka was developing theoretical models of chemical oscillations.[3] The model has a few key features. Prey, without predators, grows exponentially. Predators without prey die out. When prey and predators are present, the predators limit the growth of the prey, and the number of predators grows proportionately to the number of the prey available. The result is an oscillating pattern of population sizes.

Warning….

Fasten your seatbelts, don't let the predator knowledge get to you

The original Lotka-Volterra model itself is in the form of a differential equation given by $\frac{dN_1}{dt} = r_1 N_1 - b_1 N_1 N_2$ where r_1 is the growth of the prey population in the absence of predators, and b_1 is the predation rate by the predators, in other words, the rate at which the prey are being eaten by the predators who are represented by N_2. In words this models tells us that the rate of growth of the prey population, with respect to time, is equal to the growth rate of the prey population times the total prey population minus the predation rate times the number of prey times the number of predators. *WOW – did you get all of that!* Good, here we go again…..

The growth equation for the predator population is given by $\frac{dN_2}{dt} = -r_2 N_2 + b_2 N_2 N_1$ where r_2 is the "decay" rate of the predator population in the absence of prey (in other words the rate at which the predators are dying because there is no prey, ah, that's why it's a negative r); b_2 is the growth rate of the predator population.

[3]"Modelling Populations, From Malthus to the Threshold of Artificial Life", Mikel Maron, Project for Formal Computational Systems, University of Sussex, Dec. 2003.

Prey death and predator growth are interdependent on the size of both populations. An interesting observation: populations of predator and prey visualized as sinusoidal waves are always a one-quarter cycle out of phase, with the prey leading (which means that the peak of the prey population precedes the peak of the predator population). If you think about that for a moment it will make a lot of sense. You'll have an opportunity to apply that thinking on one of the worksheets for this unit.

When a population of any kind of animal exists in the wild, there are some things that must happen. (1) The animals get hungry, and they spend time and effort in searching for food. (2) The animals reproduce. (3) They wander to other places (migrate). (4) Sometimes their search for food means that they serve as predators to other animals in the region. (5) The animals eventually die.

What may we want to accomplish by the end of the investigations?

- What forces affect the rabbit population (positive and negative)?
- What forces affect the fox population (positive and negative)?
- When will both populations become stable (at equilibrium)?
- What does the graph of the rabbit population look like in the absence of foxes?
- What does the graph of the rabbit population look like in the presence of foxes?
- Graph rabbits vs. foxes
- Graph rabbits vs. time
- Graph foxes vs. time

Suppose we are dealing with only two species, one of which survives by preying on the other species. We will identify one species as the predator and the other as the prey. Let's call the rabbits the prey and the foxes as the predators. We want to try to see how the populations of the rabbits and the foxes change over time.

We will be doing mathematical modeling to analyze how these two populations change, so we need to look at the four major parts of a model:

I = identify the problem
S = simplify the situation
B = build a model
E = evaluate how well the model describes the actual situation

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MODEL 1:

For the initial model, the problem is to understand how a rabbit population grows. This is a very simplified model to begin. The assumptions are:

- There are no foxes.
- The rabbit population grows only by reproduction.
- Rabbits do not die.
- There is plenty of food for them to eat.
- There is no migration or other influence to increase or decrease the rabbit population.

Let's begin with a population of rabbits. Suppose that the original population of rabbits is 1000, and they grow in number by 9% per month. So the new population of rabbits equals the previous population of rabbits plus a 9% increase. This can be written as

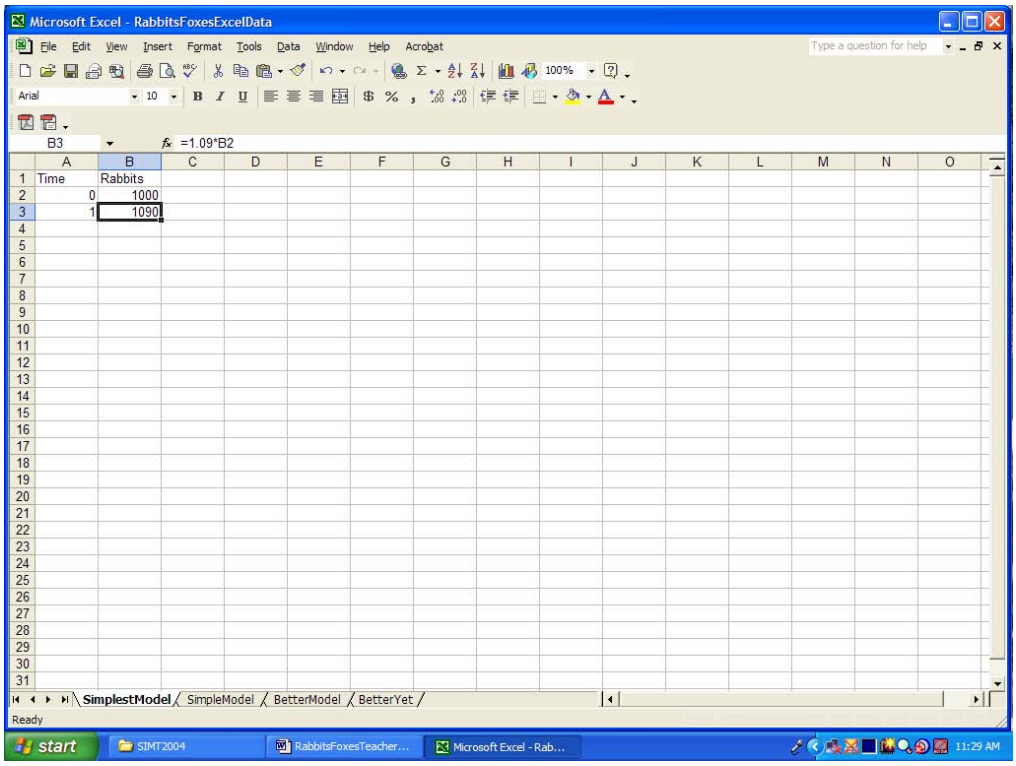
$$\text{rabbits}_{new} = \text{rabbits}_{old} + 0.09 * \text{rabbits}_{old} \qquad \text{or} \qquad R_n = R_{n-1} + 0.09\, R_{n-1}$$

These are called <u>recursively defined functions</u>. In order to find the new population, you have to know the previous population. R_n measures the new population, and R_{n-1} represents the immediately preceding month, so R_2 represents the population in month #2, and R_1 represents the rabbit population in month #1. You cannot jump to the population at the end of 6 months, for example, until you know the population for the fifth month, which means that you need the population for the fourth month, which means that you need the population for the third month, etc. You have to go back to the original population in month.

1. Use these recursive functions to fill in the following chart

Month	Rabbit population
0	1,000
1	
2	
3	
…	
n	→ the generic formula

2. We will also produce this chart on a spreadsheet. The advantage of a spreadsheet is that a lot of values can be calculated quickly and accurately, all of the values can be viewed to search for patterns, and the technology can be woven into the analysis.



To add more months, type in consecutive numbers, possibly up to 15. Then highlight the "90" and hold the CTRL-D key down to scroll down in column B to be as long as you wish.

MODEL 2:

The foxes have a problem. Suppose that there are 100 foxes that are living in the absence of rabbits. Their population is declining at a rate of 8% per month. (They do reproduce, but their numbers cannot sustain themselves because of the lack of rabbits to eat.).

3. You decide on the assumptions about the fox population:

- .
- .
- .
- .

The new population of foxes equals the previous population of foxes minus an 8% decrease. This can be written as:

4. $foxes_{new} = foxes_{old}$ - _____* $foxes_{old}$ or F_n = _____

5. Use these recursive functions to fill in the following chart

Year	Fox population
0	100
1	
2	
3	
…	
n	→ the generic formula

6. Then add a new column to the spreadsheet, headed "Foxes" to do the same thing as you did for rabbits.

7. Comment on how realistic these models are for both the rabbits and the foxes.

MODEL 3:

Now, let's put the two species together in the forest. In the presence of each other, there is the positive influence of rabbits on each other, so their population grows because they reproduce, and there is the negative influence of the foxes because the foxes eat rabbits for dinner, so the rabbit population is decreased. To develop a model for the rabbit population, in terms of the previous month's populations,

(in English) The new population of rabbits is the old population of rabbits plus the increase due to reproduction minus the number of rabbits eaten by foxes.

(in mathematics)

$$\begin{aligned} \text{rabbits}_{new} &= \text{rabbits}_{old} + \Delta\,\text{rabbits} \\ &= \text{rabbits}_{old} + \text{rabbits}_{new} - \text{rabbits}_{eaten} \\ &= \text{rabbits}_{old} + a * \text{rabbits}_{old} - b * \text{foxes}_{old} \end{aligned}$$

or … $R_n = R_{n-1} + a\,R_{n-1} - b\,F_{n-1}$

The parameter "a" is a birth rate for rabbits, and the parameter "b" is a death rate of rabbits due to their being eaten by foxes.

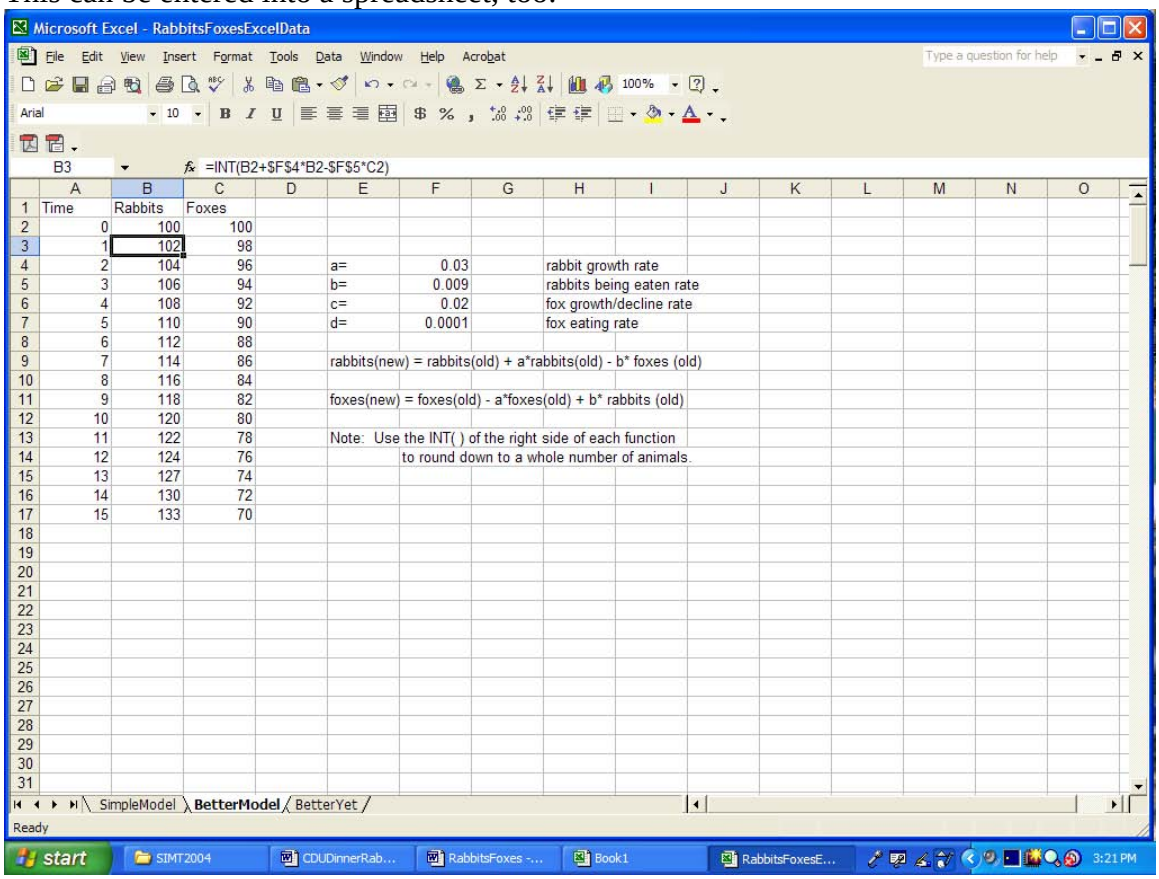
8. The foxes, on the other, hand are negatively affected by the presence of other foxes alone, because there is not enough food to survive, but they are positively affected by the

presence of rabbits. Write a model for the fox population, in terms of the previous month's populations.

(in English)

(in mathematics) $foxes_{new}$ =

This can be entered into a spreadsheet, too:



9. Enter the data and formulas. Make the time counter (which counts months) in column A rather long (by "filling down"), because the patterns, if any exist, may take a long time to become apparent.

10. Experiment with each the parameters a, b, c, and d (or whatever you called them) to see if you can make both populations survive longer. Work with one parameter at a time, and watch to see what happens at the bottom of the spreadsheet to understand the long-

term effect on the populations. Ideally, both populations should never become very large nor should either population drop to zero (or below zero!). What does a zero or negative population mean?

11. Comment on the values of the parameters that do a better job at longer survival times for the rabbits and foxes, either individually or together.

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WORKSHEET TWO

MODEL 4:

Some of the theory of populations involved in predator-prey relationships says that the number of encounters that occur between the two affects the populations. This makes sense because a predator cannot adversely affect its prey unless the two meet. The number of encounters is proportional to the product of the two populations.

$rabbits_{new} = a*rabbits_{old} – h*(rabbits_{old})(foxes_{old})$

"h" is the interaction constant between rabbits and foxes
(Why is the last term subtracted?)

1. $foxes_{new}$ =

Use a parameter called "k" for the fox population
(Which terms are negative? Which terms are positive? Why?)

2. Comment on why the product would measure the number of encounters? What happens if one population is high and the other is low? What happens if one species has become extinct? What if both populations are robust (that is, both populations are thriving and growing)?

3. Design a spreadsheet to give the populations of rabbits and foxes after several years.

4. Experiment, using trial and error, with the parameters "h" and "k" to get populations that will both survive and neither will get too large. Perhaps you should experiment with only one at a time to se what its effect is before working with the other one.

OPTIONAL CALCULATOR ACTIVITY

A table of these populations can be produced on a graphing calculator. Since the populations are modeled recursively (that is, the current population is calculated based on last month's population), then the calculator should be used in the SEQUENCE mode. (Use the MODE key, and select SEQ in the FUNC PAR POL SEQ line)

The recursively defined functions that were developed above can be put into the calculator. The "y=" key is used, but it looks different in the SEQUENCE mode. If the variable "u' represents the rabbits" and the variable "v" represents the foxes, then a possible set of functions to model the populations is shown below.

```
 Plot1 Plot2 Plot3
 nMin=0
u(n)=1.12u(n-1)
-.002v(n-1)
 u(nMin)={100}
v(n)=v(n-1)+.00
2v(n-1)+.005u(n-
1)
```

u(n) might represent (rabbits$_{new}$ or R_n), and **u(n-1)** represents (rabbits$_{old}$ or R_{n-1}). Similarly the "v" sequence functions represent the old and new fox populations.

In order to produce a graph on the screen, the WINDOW needs to be set. The WINDOW requires information about the counter range (the "n" min and max, which stands for the months which have occurred) as well as the x-min, x-max (for minimum and maximum rabbit population) and y-min and y-max (for minimum and maximum fox population).

```
WINDOW
 nMin=0
 nMax=30
 PlotStart=1
 PlotStep=1
 Xmin=0
 Xmax=200
↓Xscl=10
```

(more→)

```
WINDOW
↑PlotStep=1
 Xmin=0
 Xmax=200
 Xscl=10
 Ymin=0
 Ymax=100
 Yscl=0
```

5. Produce a graph of (rabbits, foxes). Interpret the results. Change the parameters and graph again. Explain the graphical differences in the context of the populations.

6. Produce a graph of (years, rabbits) and (years, foxes) together on the same graph. Describe why the graph looks like it does? Why does it make sense in comparison with the (rabbits, fox) graph above?

7. Which models are exponential? Which models are periodic? How can these two mathematical functions both be involved as models of population growth or decline?

8. "Sensitivity of a parameter" is a measure of how small changes in the value of the parameter can produce drastic changes in the value of the other variables? Which parameters are the most sensitive? What do they measure about the populations?

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WORKSHEET THREE

You will be working without your teacher's input on this activity. You will use the ideas that you have developed on the first two worksheets.

Look up data about the populations of a predator-prey relationship (such as whales and plankton, smokers and non-smokers, moose and bears, etc.). You will build a model to simulate their growth. But you will add another factor: Suppose that each month (or other appropriate time period), there is a flat damper on one of the populations (or both of them), such as the salinity of the water inhibits the whale and plankton population by a constant factor, or a drought on the Canadian plains causes both the moose and the bear populations to struggle to grow.

1. You will write up a report on your model.

2. Use the ISBE menu (identify the problem, simplify the problem, build the model, and evaluate the model).

3. State how you developed your model, and provide numerical and graphical support for the results.

4. Evaluate the results, and comment on how the model worked to simulate the data you found or expected, and how the model could have been changed to improve its predictions.

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WORKSHEET FOUR

Assessment

Selected Response. Provide only one answer for each of these.

1. If $A_n = 1.03A_{n-1}$, then the population
   a. declines
   b. grows
   c. oscillates
   d. is chaotic

2. Which of the following models a predator – prey relationship?

   a. $\begin{aligned} B_n &= 1.12B_{n-1} - .002B_{n-1}C_{n-1} \\ C_n &= .89C_{n-1} + .038B_{n-1}C_{n-1} \end{aligned}$

   b. $\begin{aligned} B_n &= 1.12B_{n-1} - .002B_{n-1}C_{n-1} \\ C_n &= -.89C_{n-1} + .038B_{n-1}C_{n-1} \end{aligned}$

   c. $\begin{aligned} B_n &= .93B_{n-1} - .002B_{n-1}C_{n-1} \\ C_n &= -.89C_{n-1} + .038B_{n-1}C_{n-1} \end{aligned}$

   d. $\begin{aligned} B_n &= B_{n-1} + .2B_{n-1}C_{n-1} \\ C_n &= .8C_{n-1} + .3B_{n-1}C_{n-1} \end{aligned}$

Short Constructed Response. Provide short verbal answers or calculations for each of these.

3. If $B_0 = 1000$ (prey) and $C_0 = 800$ (predator) and the population changes according to your answer to #2, evaluate B_5 and C_5 using an EXCEL spreadsheet. Attach a hard copy of the spreadsheet to this document.

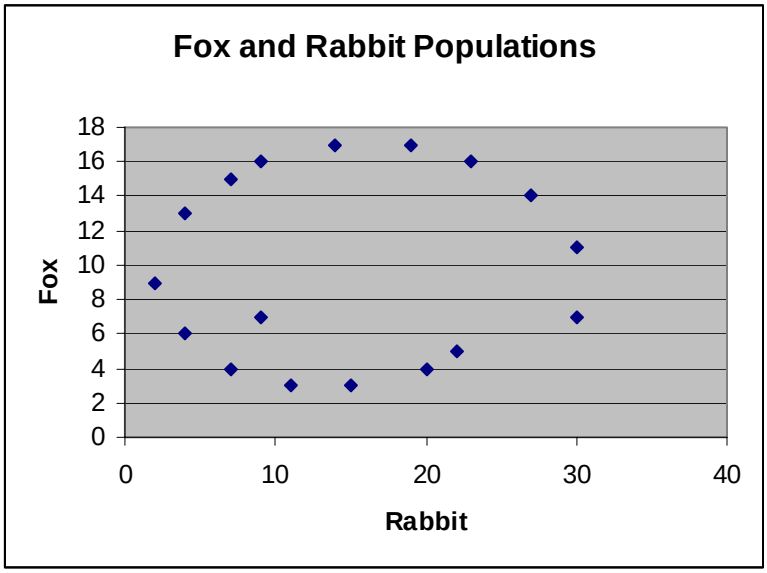
4. For $\begin{aligned} B_n &= 1.03B_{n-1} - .00001B_{n-1}C_{n-1} \\ C_n &= .87C_{n-1} + .08B_{n-1}C_{n-1} \end{aligned}$

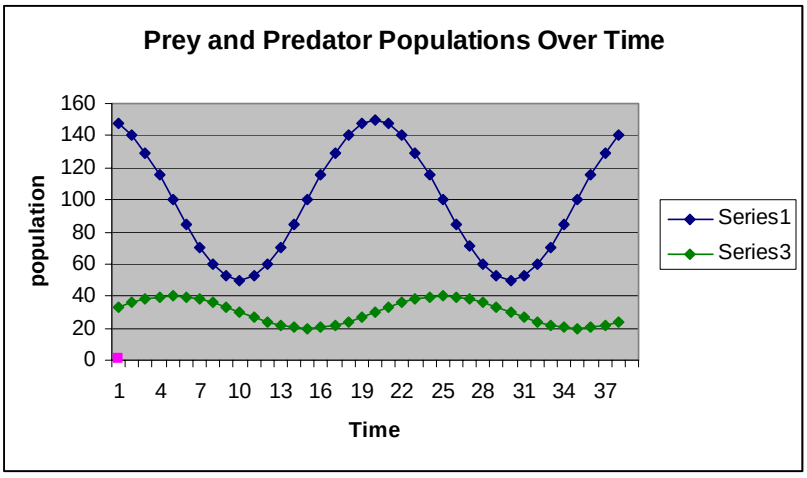
Describe what happens to B_n. Explain why this happens in terms of recursive functions.

.

5. If you visualize the population of the predator and prey as sinusoidal function why are the populations out of phase (that is, one of the periodic functions has its peak at a different time than the other function), and why does the prey lead the predator?

6. Explain what each of these graphs tells you about the populations of foxes and rabbits.





Teacher Notes and Answer Keys

Dinner at the Rabbit Café

WORKSHEET ONE

Note: The teacher notes and answer key items are all typed in italics.

When a population of any kind of animal exists in the wild, there are some things that must happen. (1) The animals get hungry, and they spend time and effort in searching for food. (2) The animals reproduce. (3) They wander to other places (migrate). (4) Sometimes their search for food means that they serve as predators to other animals in the region. (5) The animals eventually die.

What may we want to learn at the end of the investigations?

- What forces affect the rabbit population (positive and negative)?
- What forces affect the fox population (positive and negative)?
- When will both populations become stable (at equilibrium)?
- What does the graph of the rabbit population look like in the absence of foxes?
- What does the graph of the rabbit population look like in the presence of foxes?
- Graph rabbits vs. foxes
- Graph rabbits vs. time
- Graph foxes vs. time

Suppose we are dealing with only two species, one of which survives by preying on the other species. We will identify one species as the predator and the other as the prey. Let's call the rabbits the prey and the foxes as the predators. We want to try to see how the populations of the rabbits and the foxes change over time.

We will be doing mathematical modeling to analyze how these two populations change, so we need to look at the four major parts of a model:

I = identify the problem
S = simplify the situation
B = build a model
E = evaluate how well the model describes the actual situation

---

MODEL 1:

For the initial model, the problem is to understand how a rabbit population grows. This is a very simplified model to begin. The assumptions are:

- There are no foxes.
- The rabbit population grows only by reproduction.

- Rabbits do not die.
- There is plenty of food for them to eat.
- There is no migration or other influence to increase or decrease the rabbit population.

Let's begin with a population of rabbits. Suppose that the original population of rabbits is 1000, and they grow in number by 9% per month. So the new population of rabbits equals the previous population of rabbits plus a 9% increase. This can be written as

$$\text{rabbits}_{new} = \text{rabbits}_{old} + 0.09^* \text{ rabbits}_{old} \qquad \text{or} \qquad R_n = R_{n-1} + 0.09\, R_{n-1}$$

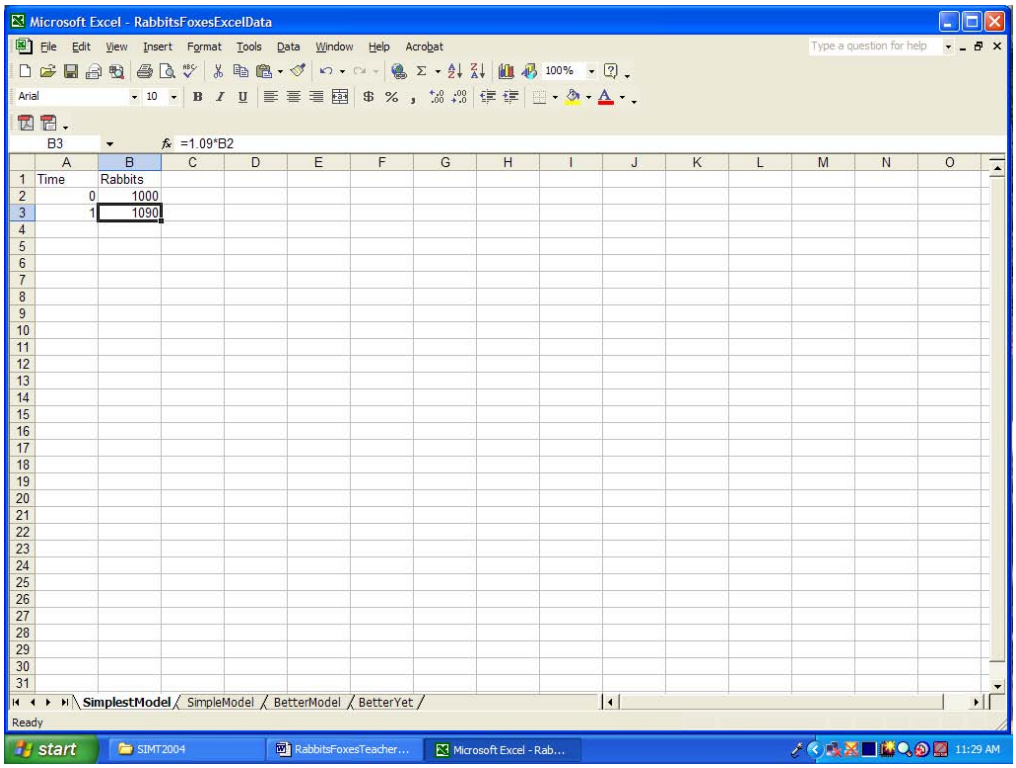
These are called <u>recursively defined functions</u>. In order to find the new population, you have to know the previous population. R_n measures the new population, and R_{n-1} represents the immediately preceding month, so R_2 represents the population in month #2, and R_1 represents the rabbit population in month #1. You cannot jump to the population at the end of 6 months, for example, until you know the population for the fifth month, which means that you need the population for the fourth month, which means that you need the population for the third month, etc. You have to go back to the original population in month.

1. Use these recursive functions to fill in the following chart

Month	Rabbit population
0	1,000
1	*1090*
2	*1188*
3	*1295*
…	
n	*1.09n* → the generic formula

*To do this recursively on the TI-83 calculator, type in 1000 and hit ENTER. Then type ANS (above the "-" key) + .09*ANS (or 1.09* ANS). If you then hit the ENTER key over and over, you will get a sequence of the rabbit populations, 1000, 1090, 1188, etc.*

2.We will also produce this chart on a spreadsheet.



To add more months, type in consecutive numbers, possibly up to 15. Then highlight the "90" and hold the CTRL-D key down to scroll down in column B to be as long as you wish.

MODEL 2:

The foxes have a problem. Suppose that there are 100 foxes that are living in the absence of rabbits. Their population is declining at a rate of 8% per month. (They do reproduce, but their numbers cannot sustain themselves because of the lack of rabbits to eat.).

3. You decide on the assumptions about the fox population:
    - *There are no rabbits to eat.*
    - *The fox population grows only by reproduction.*
    - *Foxes do not die.*
    - *There is no migration or other influence to increase or decrease the fox population.*

The new population of foxes equals the previous population of foxes minus an 8% decrease. This can be written as:

4. $\text{foxes}_{new} = \text{foxes}_{old} - 0.08 * \text{foxes}_{old}$ or $F_n = F_{n-1} - 0.08 * F_{n-1}$

5. Use these recursive functions to fill in the following chart

Year	Fox population
0	100
1	*92*
2	*84 (round down!)*
3	*77*
…	
n	*0.92n* → the generic formula

This can be done on the calculator, with "100" as the initial value, and ANS.92 done repeatedly to get the subsequent values, just as was done above.*

6. Then add a new column to the spreadsheet, headed as "Foxes" to do the same thing as you did for rabbits.

Microsoft Excel - RabbitsFoxesExcelData

File Edit View Insert Format Tools Data Window Help Acrobat

G7 =INT(1.09*G6)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	Time	Rabbits	Foxes												
2	0	1000	100		Make the populations integers with the int( ) function										
3	1	1090	92		(which rounds down)										
4	2	1188.1	84.64												
5	3	1295.029	77.8688			Time	Rabbits	Foxes							
6	4	1411.582	71.6393			0	1000	100							
7	5	1538.624	65.90815			1	1090	92							
8	6	1677.1	60.6355			2	1188	84							
9	7	1828.039	55.78466			3	1294	77							
10	8	1992.563	51.32189			4	1410	70							
11	9	2171.893	47.21614			5	1536	64							
12	10	2367.364	43.43885			6	1674	58							
13	11	2580.426	39.96374			7	1824	53							
14	12	2812.665	36.76664			8	1988	48							
15	13	3065.805	33.82531			9	2166	44							
16	14	3341.727	31.11928			10	2360	40							
17	15	3642.482	28.62974			11	2572	36							
18	16	3970.306	26.33936			12	2803	33							
19	17	4327.633	24.23221			13	3055	30							
20	18	4717.12	22.29364			14	3329	27							
21	19	5141.661	20.51014			15	3628	24							
22	20	5604.411	18.86933			16	3954	22							
23	21	6108.808	17.35979			17	4309	20							
24	22	6658.6	15.971			18	4696	18							
25	23	7257.874	14.69332			19	5118	16							
26	24	7911.083	13.51786			20	5578	14							
27						21	6080	12							
28						22	6627	11							
29						23	7223	10							
30						24	7873	9							
31															

SimplestModel / SimpleModel / BetterModel / BetterYet

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7. Comment on how realistic these models are for both the rabbits and the foxes.

Since the rabbits have no predators or anything else to slow their population growth, and since the foxes have no food source, these models are not realistic. If they were, we would already be overrun with rabbits and would never know what a fox was, since it would be long extinct.

MODEL 3:

Now, let's put the two species together in the forest. In the presence of each other, there is the positive influence of rabbits on each other, so their population grows because they reproduce, and there is the negative influence of the foxes because the foxes eat rabbits for dinner, so the rabbit population is decreased. To develop a model for the rabbit population, in terms of the previous month's populations,

(in English) The new population of rabbits is the old population of rabbits plus the increase due to reproduction minus the number of rabbits eaten by foxes.

(in mathematics)

$$\begin{aligned} \text{rabbits}_{new} &= \text{rabbits}_{old} + \Delta\,\text{rabbits} \\ &= \text{rabbits}_{old} + \text{rabbits}_{new} - \text{rabbits}_{eaten} \\ &= \text{rabbits}_{old} + a * \text{rabbits}_{old} - b * \text{foxes}_{old} \end{aligned}$$

or … $R_n = R_{n-1} + a\,R_{n-1} - b\,F_{n-1}$

The parameter "a" is a birth rate for rabbits, and the parameter "b" is a death rate of rabbits due to their being eaten by foxes.

The foxes, on the other, hand are negatively affected by the presence of other foxes alone, because there is not enough food to survive, but they are positively affected by the presence of rabbits. Write a model for the fox population, in terms of the previous month's populations.

8. (in English) *The new population of foxes is the old population of foxes minus a percentage due to reproduction (because of the increased number of foxes to be fed) plus the number of foxes who can survive by eating rabbits.*

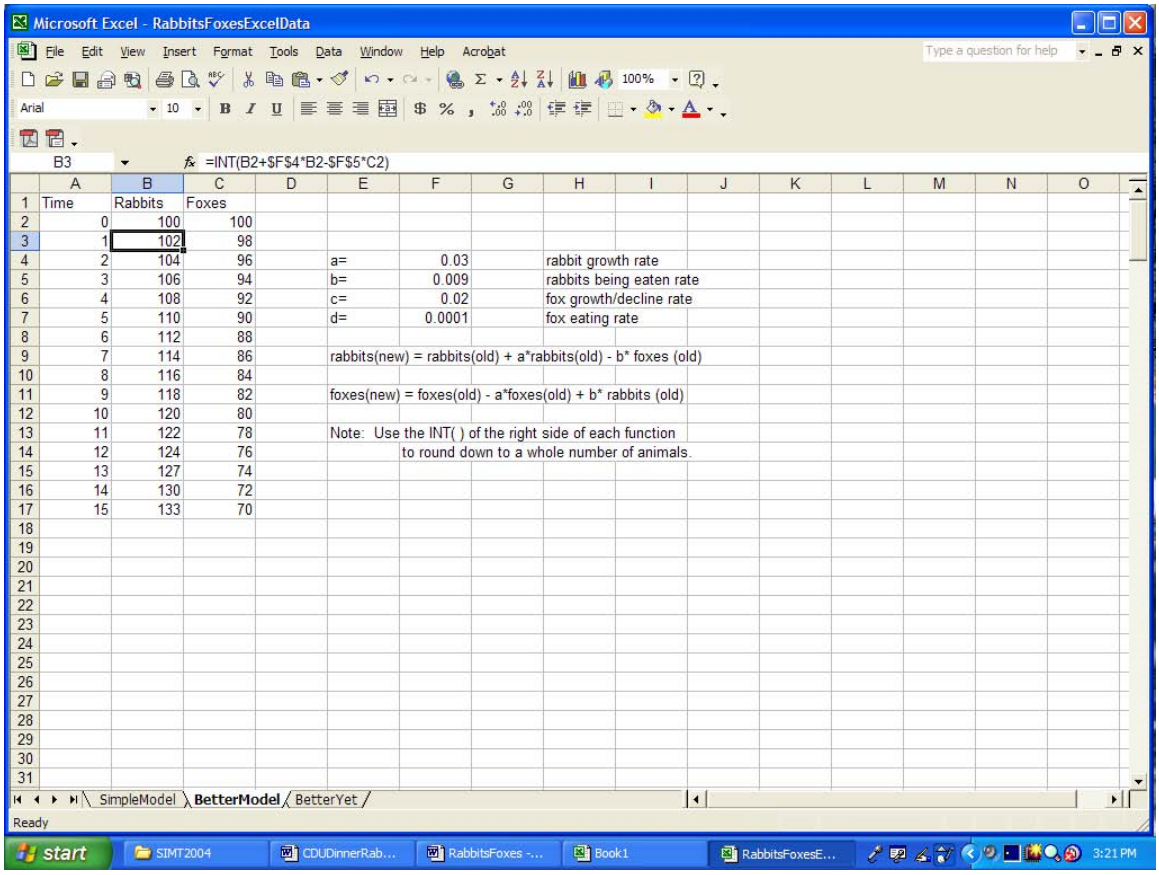
(in mathematics)

$$\begin{aligned} foxes_{new} &= foxes_{old} + \Delta\,foxes \\ &= foxes_{old} + foxes_{new} + rabbits_{eaten} \\ &= foxes_{old} + a * foxes_{old} - b * rabbits_{old} \end{aligned}$$

or … $F_n = F_{n-1} - c\,F_{n-1} + d\,R_{n-1}$

where "c" is the birth rate for foxes, and "d" is the rate at which rabbits are eaten.

This can be entered into a spreadsheet, too:



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	Time	Rabbits	Foxes												
2	0	100	100												
3	1	102	98												
4	2	104	96		a=	0.03		rabbit growth rate							
5	3	106	94		b=	0.009		rabbits being eaten rate							
6	4	108	92		c=	0.02		fox growth/decline rate							
7	5	110	90		d=	0.0001		fox eating rate							
8	6	112	88												
9	7	114	86		rabbits(new) = rabbits(old) + a*rabbits(old) - b* foxes (old)										
10	8	116	84												
11	9	118	82		foxes(new) = foxes(old) - a*foxes(old) + b* rabbits (old)										
12	10	120	80												
13	11	122	78		Note: Use the INT( ) of the right side of each function										
14	12	124	76			to round down to a whole number of animals.									
15	13	127	74												
16	14	130	72												
17	15	133	70												

9. Enter the data and formulas. Make the time counter (which counts months) in column A rather long (by "filling down"), because the patterns, if any exist, may take a long time to become apparent.

10. Experiment with each the parameters a, b, c, and d (or whatever you called them) to see if you can make both populations survive longer. Work with one parameter at a time, and watch to see what happens at the bottom of the spreadsheet to understand the long-term effect on the populations. Ideally, each population should never become very large nor should either population drop to zero (or below zero!). What does a zero or negative population mean?

11. Comment on the values of the parameters that do a better job at longer survival times for the rabbits and foxes, either individually or together.

There is a lot of sensitivity to the parameters, but some suggestions to use include:

$a = 0.03$	$c = 0.02$
$b = 0.009$	$d = 0.0001$

Changing any of these, including the initial populations leads to some interesting results. The fact that some of these kinds of numbers actually work comes from field observations of biologists.

Dinner at the Rabbit Café

WORKSHEET TWO

MODEL 4:

Some of the theory of populations involved in predator-prey relationships says that the number of encounters that occur between the two affects the populations. This makes sense because a predator cannot adversely affect its prey unless the two meet. The number of encounters is proportional to the product of the two populations.

$rabbits_{new} = a*rabbits_{old} – h*(rabbits_{old})(foxes_{old})$

"h" is the interaction constant between rabbits and foxes
(Why is the last term subtracted?)

The last term is subtracted because the interactions between rabbits and foxes are harmful to the rabbit population.

1. $foxes_{new} = -c*foxes_{old} + k*(rabbits_{old})(foxes_{old})$

Use a parameter called "k" for the fox population
(Which terms are negative? Which terms are positive? Why?)

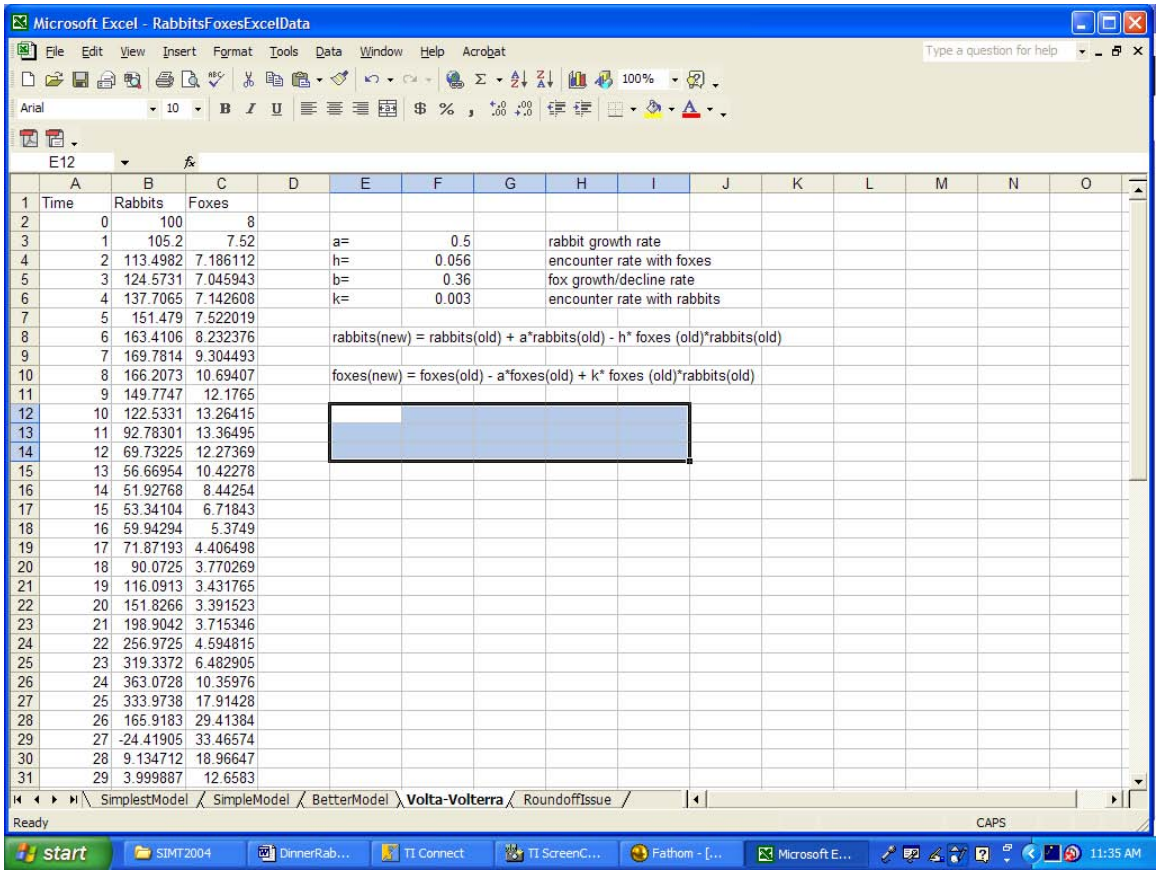
The first term is negative, because the presence of more foxes makes survival harder for all of the foxes (although breeding is good) as they compete for food, and the second term is positive because the interaction with rabbits is great for the fox population, given that rabbits are the foods source.

2. Comment on why the product would measure the number of encounters? What happens if one population is high and the other is low? What happens if one species has become extinct? What if both populations are robust (that is, both populations are thriving and growing)?

The product is affected by the size of both populations, so the product is high only if both populations are large meaning that there is more opportunity for the two populations to interact (even probabilistically). If one population is high and the other is low, the product will be low, and the number of interactions is low. If one population becomes extinct, then the product is zero, and the predator's population will then become zero eventually too. This says that the model is working well, since that is what will actually happen. If both populations are high, then there will be a lot of interactions; the foxes

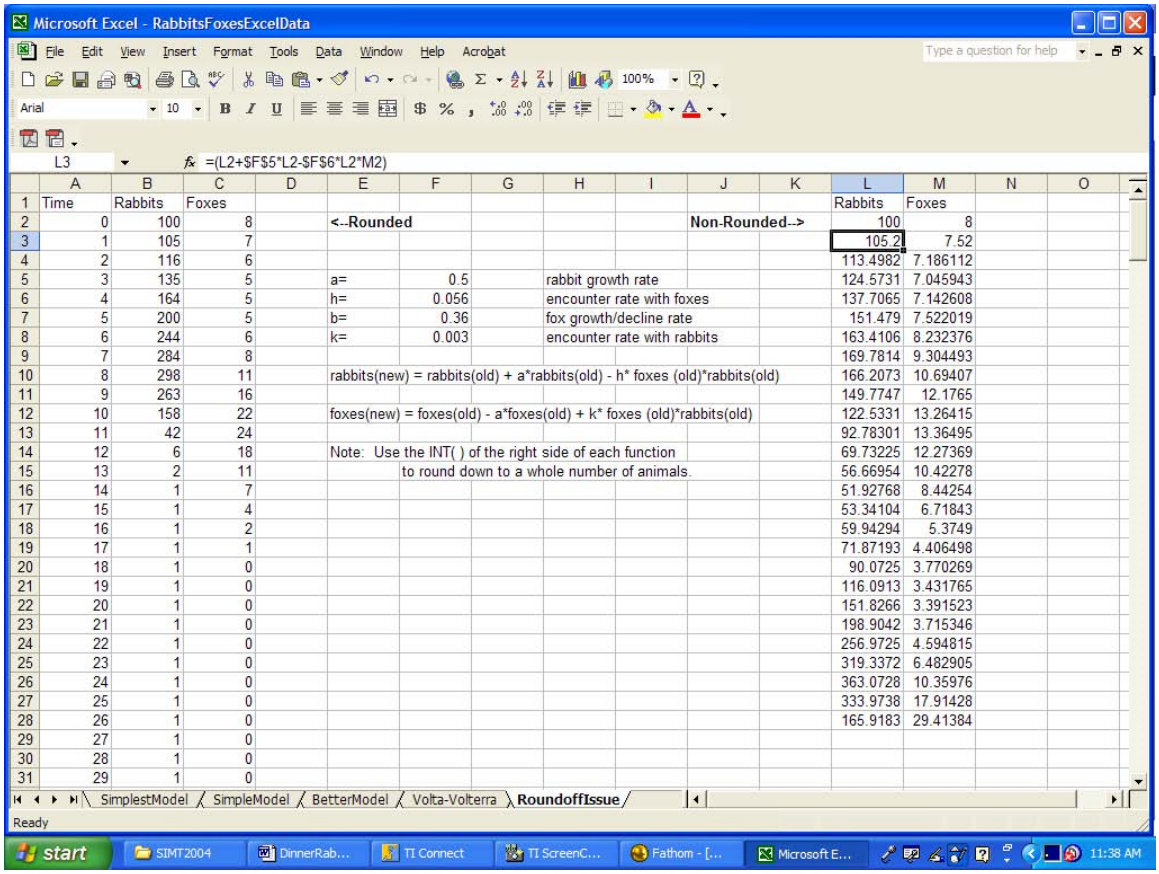
will eat well, rabbits will be eaten, but there are a lot of rabbits available, at least for a while.

1. Design a spreadsheet to give the populations of rabbits and foxes after several years.



	A	B	C	D	E	F	G	H	I
1	Time	Rabbits	Foxes						
2	0	100	8						
3	1	105.2	7.52		a=	0.5		rabbit growth rate	
4	2	113.4982	7.186112		h=	0.056		encounter rate with foxes	
5	3	124.5731	7.045943		b=	0.36		fox growth/decline rate	
6	4	137.7065	7.142608		k=	0.003		encounter rate with rabbits	
7	5	151.479	7.522019						
8	6	163.4106	8.232376		rabbits(new) = rabbits(old) + a*rabbits(old) - h* foxes (old)*rabbits(old)				
9	7	169.7814	9.304493						
10	8	166.2073	10.69407		foxes(new) = foxes(old) - a*foxes(old) + k* foxes (old)*rabbits(old)				
11	9	149.7747	12.1765						
12	10	122.5331	13.26415						
13	11	92.78301	13.36495						
14	12	69.73225	12.27369						
15	13	56.66954	10.42278						
16	14	51.92768	8.44254						
17	15	53.34104	6.71843						
18	16	59.94294	5.3749						
19	17	71.87193	4.406498						
20	18	90.0725	3.770269						
21	19	116.0913	3.431765						
22	20	151.8266	3.391523						
23	21	198.9042	3.715346						
24	22	256.9725	4.594815						
25	23	319.3372	6.482905						
26	24	363.0728	10.35976						
27	25	333.9738	17.91428						
28	26	165.9183	29.41384						
29	27	-24.41905	33.46574						
30	28	9.134712	18.96647						
31	29	3.999887	12.6583						

Here is a good mathematical lesson! If you decide to round down to integer number of animals (which seems reasonable at first glance!), then the café will have to close! You will get vastly different numbers of rabbits and foxes, due to the roundoff and NOT to the fact that the model is bad. This is a valuable lesson for students to see. A spreadsheet below shows the same calculations with and without the round off.



Microsoft Excel - RabbitsFoxesExcelData

L3 =(L2+F5*L2-F6*L2*M2)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	Time	Rabbits	Foxes									Rabbits	Foxes		
2	0	100	8		<--Rounded					Non-Rounded-->		100	8		
3	1	105	7									105.2	7.52		
4	2	116	6									113.4982	7.186112		
5	3	135	5		a=	0.5		rabbit growth rate				124.5731	7.045943		
6	4	164	5		h=	0.056		encounter rate with foxes				137.7065	7.142608		
7	5	200	5		b=	0.36		fox growth/decline rate				151.479	7.522019		
8	6	244	6		k=	0.003		encounter rate with rabbits				163.4106	8.232376		
9	7	284	8									169.7814	9.304493		
10	8	298	11		rabbits(new) = rabbits(old) + a*rabbits(old) - h* foxes (old)*rabbits(old)							166.2073	10.69407		
11	9	263	16									149.7747	12.1765		
12	10	158	22		foxes(new) = foxes(old) - a*foxes(old) + k* foxes (old)*rabbits(old)							122.5331	13.26415		
13	11	42	24									92.78301	13.36495		
14	12	6	18		Note: Use the INT( ) of the right side of each function							69.73225	12.27369		
15	13	2	11			to round down to a whole number of animals.						56.66954	10.42278		
16	14	1	7									51.92768	8.44254		
17	15	1	4									53.34104	6.71843		
18	16	1	2									59.94294	5.3749		
19	17	1	1									71.87193	4.406498		
20	18	1	0									90.0725	3.770269		
21	19	1	0									116.0913	3.431765		
22	20	1	0									151.8266	3.391523		
23	21	1	0									198.9042	3.715346		
24	22	1	0									256.9725	4.594815		
25	23	1	0									319.3372	6.482905		
26	24	1	0									363.0728	10.35976		
27	25	1	0									333.9738	17.91428		
28	26	1	0									165.9183	29.41384		
29	27	1	0												
30	28	1	0												
31	29	1	0												

SimplestModel / SimpleModel / BetterModel / Volta-Volterra / RoundoffIssue

4. Experiment, using trial and error, with the parameters “h” and “k” to get populations that will both survive and neither will get too large. Perhaps you should experiment with only one at a time to see what its effect is before working with the other one.

One suggested set is a = 0.3 , h = 0.056 for the rabbits, and b = 0.36 and k = 0.003 for the foxes. (See the spreadsheet above.) Another set is a = 1.5 and h = 0.034 for rabbits, and b = 1.4 and k = .036 for foxes. An interesting question to discover why there are multiple answers, whether there are more, and how they are related to each other.

OPTIONAL CALCULATOR ACTIVITY

A table of these populations can be produced on a graphing calculator. Since the populations are modeled recursively (that is, the current population is calculated based on last month's population), then the calculator should be used in the SEQUENCE mode. (Use the MODE key, and select SEQ in the FUNC PAR POL SEQ line)

The recursively defined functions that were developed above can be put into the calculator. The "y=" key is used, but it looks different in the SEQUENCE mode. If the variable "u' represents the rabbits" and the variable "v" represents the foxes, then a possible set of functions to model the populations is shown below.

```
 Plot1 Plot2 Plot3
 nMin=0
u(n)=1.12u(n-1)
-.002v(n-1)
 u(nMin)={100}
v(n)=v(n-1)+.00
2v(n-1)+.005u(n-
1)
```

u(n) might represent ($rabbits_{new}$ or R_n), and **u(n-1)** represents ($rabbits_{old}$ or R_{n-1}). Similarly the "v" sequence functions represent the old and new fox populations.

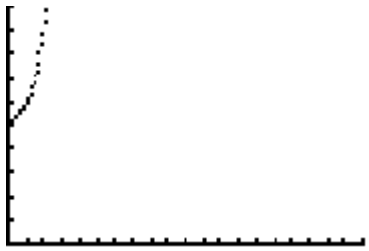
In order to produce a graph on the screen, the WINDOW needs to be set. The WINDOW requires information about the counter range (the "n" min and max, which stands for the months which have occurred) as well as the x-min, x-max (for minimum and maximum rabbit population) and y-min and y-max (for minimum and maximum fox population).

```
WINDOW
 nMin=0
 nMax=30
 PlotStart=1
 PlotStep=1
 Xmin=0
 Xmax=200
↓Xscl=10
```

(more→)

```
WINDOW
↑PlotStep=1
 Xmin=0
 Xmax=200
 Xscl=10
 Ymin=0
 Ymax=100
 Yscl=0
```

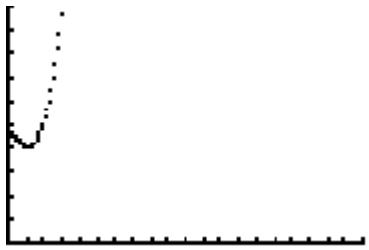
5. Produce a graph of (rabbits, foxes). Interpret the results. Change the parameters and graph again. Explain the graphical differences in the context of the populations.



In this case, both populations are growing without bound.

Following is a second set of recursive functions and their graph, using SEQUENCE mode.

```
Plot1 Plot2 Plot3
nMin=0
u(n)=1.12u(n-1)
-.002v(n-1)
u(nMin)={100}
v(n)=.96v(n-1)+
.005u(n-1)
v(nMin)={50}
```



6. Produce a graph of (years, rabbits) and (years, foxes) together on the same graph. Describe why the graph looks like it does? Why does it make sense in comparison with the (rabbits, fox) graph above?

These are the recursive functions from which graphs could be produced. The actual graphs do not appear to be periodic on the calculator, but the data seems to be sinusoidal.

```
Plot1 Plot2 Plot3
nMin=0
u(n)=1.02u(n-1)
-.462v(n-1)
u(nMin)={100}
v(n)=-.91v(n-1)
+.28u(n-1)
v(nMin)={70}
```

7. Which models are exponential? Which models are periodic? How can these two mathematical functions both be involved as models of population growth or decline?

In the absence of a predator, the rabbit population is exponential. Under certain conditions (with specific parameters), the populations of each of the species will cycle, so the models are periodic.

8. "Sensitivity of a parameter" is a measure of how small changes in the value of the parameter can produce drastic changes in the value of the other variables? Which parameters are the most sensitive? What do they measure about the populations?

There are four parameters to change, so there are lots of possible conclusions and interpretations.

Dinner at the Rabbit Café

WORKSHEET THREE

You will be working without your teacher's input on this activity. You will use the ideas that you have developed on the first two worksheets.

Look up data about the populations of a predator-prey relationship (such as whales and plankton, smokers and non-smokers, moose and bears, etc.). You will build a model to simulate their growth. But you will add another factor: Suppose that each month (or other appropriate time period), there is a flat damper on one of the populations (or both of them), such as the salinity of the water inhibits the whale and plankton population by a constant factor, or a drought on the Canadian plains causes both the moose and the bear populations to struggle to grow.

1. You will write up a report on your model.

2. Use the ISBE menu (identify the problem, simplify the problem, build the model, and evaluate the model).

3. State how you developed your model, and provide numerical and graphical support for the results.

4. Evaluate the results, and comment on how the model worked to simulate the data you found or expected, and how the model could have been changed to improve its predictions.

Dinner At The Rabbit Café

WORKSHEET FOUR

Assessment

Selected Response. Provide only one answer for each of these.

1. If $A_n = 1.03A_{n-1}$, then the population
   a. declines
   b. grows
   c. oscillates
   d. is chaotic

2. Which of the following best models a predator – prey relationship?

   a. $\begin{aligned} B_n &= 1.12B_{n-1} - .002B_{n-1}C_{n-1} \\ C_n &= .89C_{n-1} + .038B_{n-1}C_{n-1} \end{aligned}$

   b. $\begin{aligned} B_n &= 1.12B_{n-1} - .002B_{n-1}C_{n-1} \\ C_n &= -.89C_{n-1} + .038B_{n-1}C_{n-1} \end{aligned}$

   c. $\begin{aligned} B_n &= .93B_{n-1} - .002B_{n-1}C_{n-1} \\ C_n &= -.89C_{n-1} + .038B_{n-1}C_{n-1} \end{aligned}$

   d. $\begin{aligned} B_n &= B_{n-1} + .2B_{n-1}C_{n-1} \\ C_n &= .8C_{n-1} + .3B_{n-1}C_{n-1} \end{aligned}$

Short Constructed Response. Provide short verbal answers or calculations for each of these.

3. If $B_0 = 1000$ (prey) and $C_0 = 800$ (predator) and the population changes according to your answer to #2, evaluate B_5 and C_5 using an EXCEL spreadsheet. Attach a hard copy of the spreadsheet to this document.

4. For $\begin{aligned} B_n &= 1.03B_{n-1} - .00001B_{n-1}C_{n-1} \\ C_n &= .87C_{n-1} + .08B_{n-1}C_{n-1} \end{aligned}$

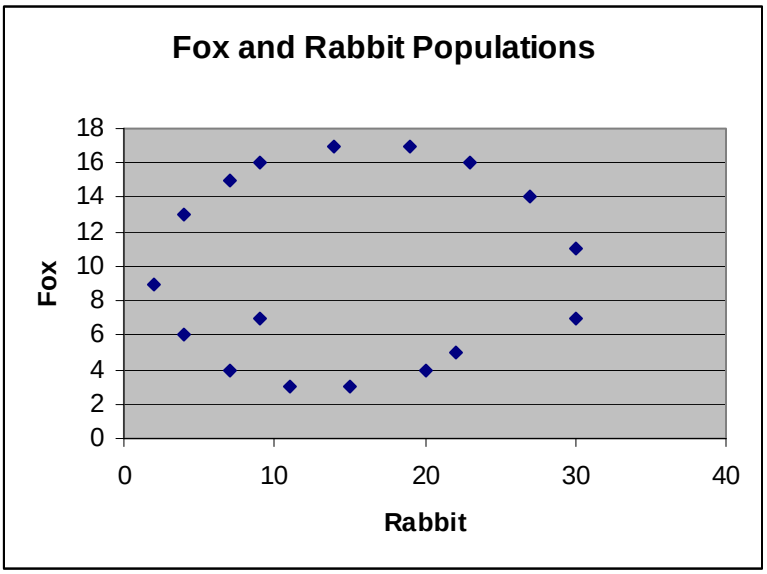
Describe what happens to B_n. Explain why this happens in terms of recursive functions.

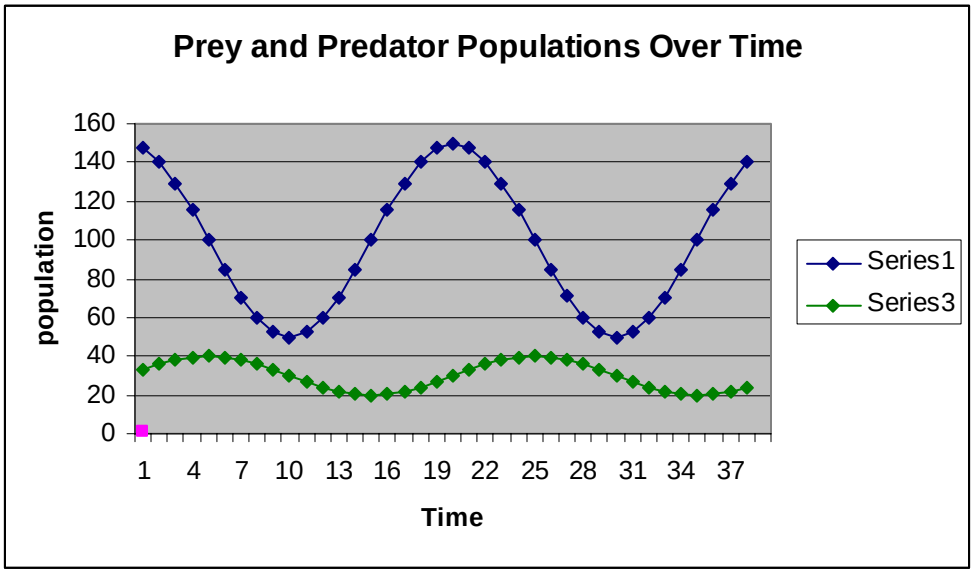
Prey B_n will grow, because its growth rate is 103% and the interaction rate with its predator is small (-.00001).

5. If you visualize the population of the predator and prey as sinusoidal functions, why are the populations out of phase (that is, one of the periodic functions has its peak at a different time than the other function), and why does the prey lead the predator?

The predator is out of phase with the prey, because it takes time for the predator to realize that there is less prey available to eat, and the prey has to decline before the predator will, given that the prey is the food source.

6. Explain what each of these graphs tell you about the populations of foxes and rabbits





This graph says that the prey and its predator are in equilibrium, since they cycle in an ellipse shape. The rabbit population will grow while the fox population grows until three are enough foxes to eat rabbits and force the rabbit population to decline (at the right side of the ellipse). After a one-quarter cycle, the fox population is at its largest, and the rabbit population is declining. The foxes then begin to decline, as their food supply is smaller. After another one-quarter cycle, the fox population is declining, and the rabbit finally begin to recover and begin to increase again, since there are fewer foxes to eat them. Another one-quarter cycle later finds the foxes at a minimum population, and the rabbits are really beginning to thrive. The fox population rebounds, and the cycle begin again.