Title: Matchstick Math: Using Manipulatives to Model Linear, Quadratic, and Exponential Functions

Brief Overview:

Students will use matchsticks and straws to create geometric patterns. Students will use the properties of these patterns (perimeter, area, total number of segments) to model linear, quadratic, and exponential functions, respectively. Students will make predictions based on the models used.

NCTM Standards

• Content Standards
  
  o Algebra

  - Understand patterns, relations, and functions

  • Generalize patterns using explicitly defined and recursively defined functions;
  • Understand relations and functions and select, convert flexibly among, and use various representations for them;
  • Analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior;
  • Understand and compare the properties of classes of functions, including exponential, polynomial, rational, logarithmic, and periodic functions;

  - Use mathematical models to represent and understand quantitative relationships

  • Identify essential quantitative relationships in a situation and determine the class or classes of functions that might model the relationships;
  • Use symbolic expressions, including iterative and recursive forms, to represent relationships arising from various contexts;
  • Draw reasonable conclusions about a situation being modeled.

  o Data Analysis and Probability

  - Select and use appropriate statistical methods to analyze data

  • For bivariate measurement data, be able to display a scatterplot, describe its shape, and determine regression coefficients, regression equations, and correlation coefficients using technological tools;
• Display and discuss bivariate data where at least one variable is categorical;
• Identify trends in bivariate data and find functions that model the data or transform the data so that they can be modeled.

• Process Standards
  o Problem Solving
    • Build new mathematical knowledge through problem solving;
    • Solve problems that arise in mathematics and in other contexts;
    • Apply and adapt a variety of appropriate strategies to solve problems;
    • Monitor and reflect on the process of mathematical problem solving.
  o Communication
    • Organize and consolidate their mathematical thinking through communication;
    • Communicate their mathematical thinking coherently and clearly to peers, teachers, and others;
    • Analyze and evaluate the mathematical thinking and strategies of others;
    • Use the language of mathematics to express mathematical ideas precisely.
  o Connections
    • Recognize and use connections among mathematical ideas;
    • Understand how mathematical ideas interconnect and build on one another to produce a coherent whole;
    • Recognize and apply mathematics in contexts outside of mathematics.
  o Representation
    • Create and use representations to organize, record, and communicate; mathematical ideas
    • Select, apply, and translate among mathematical representations to solve problems;
    • Use representations to model and interpret physical, social, and mathematical phenomena.

Grade/Level:

The target grade level is Grade 9 (Algebra I).

Duration/Length:
Each lesson is designed for a 45 minute time period.
Student Outcomes:

Lesson 1

Students will:

- Create scatter plots of data collected from matchstick shape patterns.
- Use the linear regression capabilities of the TI-83 Plus to find linear functions that model the properties of matchstick shape patterns.
- Recognize the properties of the differences between successive data points for linear functions.

Lesson 2

Students will:

- Create scatter plots of data collected from matchstick shape patterns.
- Use the quadratic regression capabilities of the TI-83 Plus to find quadratic functions that model the properties of matchstick shape patterns.
- Recognize the properties of the differences between successive data points for quadratic functions.

Lesson 3

Students will:

- Create scatter plots of data collected from tree fractal shape patterns.
- Use the exponential regression capabilities of the TI-83 Plus to find exponential functions that model the properties of tree fractal shape patterns.
- Use the Interactive Graphing Application on the TI-83 Plus to find exponential functions that model the properties of tree fractal shape patterns.
- Recognize the properties of the differences between successive data points for exponential functions.

Materials and Resources:

- Matchsticks or Toothpicks (Lessons 1 and 2)
- Straws (Lesson 3)
- TI-83 Plus Graphing Calculators
- Activity Sheets
Development/Procedures:

Lesson 1

Prerequisite Knowledge – Students should be familiar with

- The definition of a function
- Dependent and independent variables
- Graphing a line, both by hand and on graphing calculators
- Slope
- Perimeter

Preassessment – Ask students if they know the definition of a function. Students should know that a function is a rule or relationship in which there is exactly one output value for each input value.

Then review the concept of slope with students. Students should be aware that slope is represented by the variable \( m \) in the linear equation \( y = mx + b \). For example, ask students what is the slope of the line \( y = 3x - 7 \), how they determined this, and what this tells us about the graph. (The slope is 3 because 3 is the coefficient of \( x \), and this tells us that the graph is increasing from left to right).

Launch – Demonstrate to students that they will be making patterns with matchsticks. Show them the first two or three steps in the pattern shown in the Notes handout. Make sure that students are able to find the perimeters of square or rectangular geometric figures.

Teacher Facilitation – The teacher should distribute the Notes handout to the students and explain the concepts. The teacher should also circulate around the room to assist students and to monitor their progress.

Student Application – Allow the students to complete the Worksheet in cooperative groups.

Embedded Assessment – Assessment questions are included in Worksheet.

Reteaching/Extension –

- Direct a class discussion in which groups share their answers with the class. The teacher should assist any students who do not understand any of the concepts. A key is included.
- The last question of the Worksheet is an extension question.
Summary questions are included.

Summary Questions-

Here are some questions to ask the students to summarize the lesson.

1) What is a linear function?

   A linear function is a function in which successive output values increase at a constant rate.

2) How is this constant rate related to the appearance of the graph of the linear function?

   If the function’s output values increase at a constant rate, the slope of the graph is positive, and the graph is increasing from left to right.

   If the function’s output values decrease at a constant rate, the slope of the graph is negative, and the graph is decreasing from left to right.

   If the function’s output values are constant, the slope of the graph is zero, and the graph is horizontal.

3) In this lesson, our functions were always increasing. How would our pattern of shapes have to look in order for the linear function to be decreasing?

   The pattern would have to be reversed. In other words, the second step would have to use fewer matchsticks than the first step, and so on.
Notes

Linear Functions in Matchstick Patterns  Name:_______________________

The four shapes pictured below represent the first four shapes in a pattern, where the value of \( n \) represents the step in the pattern.

*Use your matchsticks to create each of these four shapes.*

One may use function notation to represent a given shape property for each step. Here, we will let the function \( f \) represent the perimeter of the largest square in each step, where the smallest square has sides of length 1 unit. The perimeter of the largest (and only) square in the first step is 4 units, thus one may write that \( f(1)=4 \). Similarly, the perimeter of the largest square in the second step is 8 units, so \( f(2)=8 \). Likewise, \( f(3)=12 \), \( f(4)=16 \), and so on.

Use your calculator to enter the input and output values for function \( f \) in a statistics table. To do this, press \[STAT\] and then select EDIT. Put the input values in the column under L1 and the output values in the column under L2. Your display should look like this:

<table>
<thead>
<tr>
<th>n=1</th>
<th>n=2</th>
<th>n=3</th>
<th>n=4</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1</td>
<td>L2</td>
<td>L3</td>
<td>L1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>8</td>
<td>----</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td>8</td>
<td>16</td>
</tr>
<tr>
<td>----</td>
<td>----</td>
<td>----</td>
<td>----</td>
</tr>
</tbody>
</table>

Note that the differences in successive output values, L2, are constant. That is, \( 8 - 4 = 4 \), \( 12 - 8 = 4 \), and \( 16 - 12 = 4 \).

Because the differences in successive output values are constant as the input value increases at a constant rate, this indicates that \( f \) is a **linear function**.

Next, we will create a scatter plot of this function.
To create a scatter plot of the function \( f \), press \( \text{WINDOW} \) and enter the values shown below:

\[
\begin{align*}
\text{WINDOW} \\
X_{\text{min}} &= 0 \\
X_{\text{max}} &= 5 \\
X_{\text{scl}} &= 1 \\
Y_{\text{min}} &= 0 \\
Y_{\text{max}} &= 20 \\
Y_{\text{scl}} &= 5 \\
X_{\text{res}} &= 1
\end{align*}
\]

Now press \( \text{2nd} \) \( \text{STAT PLOT} \) and turn on Plot 1. Then press “1” and be sure that your calculator screen looks like this:

Now, we will use linear regression to determine the equation of a line that best fits this data set. To do this, press \( \text{STAT} \), then choose \( \text{CALC} \), then choose \( \text{LinReg(ax+b)} \). Press \( \text{ENTER} \) and then press \( \text{VARS} \), and choose \( \text{Y-VARS} \) and then \( \text{Y1} \), so that your screen says \( \text{LinReg(ax+b)} \ \text{Y1} \). This will store the regression equation in \( \text{Y1} \). Press \( \text{ENTER} \).

Then, after you press \( \text{GRAPH} \), you should get the following graph:

Note that the line of best fit passes through all four points. Thus, the function is linear.

If you press \( \text{Y=} \), you will see that the equation for \( \text{Y1}=4X \). Thus, the slope of the line is equal to 4, which is the same as the constant difference in successive output values.

The equation \( y = 4x \) may be written as \( f(n) = 4n \) in function notation. These notations are interchangeable as long as the relation for \( y \) in terms of \( x \) is a function – that is, there is exactly one output value for each input value.
Worksheet
Linear Functions in Matchstick Patterns
Name: _______________________

Use your matchsticks to create the following stairstep pattern.

1) Determine the perimeter of the pattern for each step.

<table>
<thead>
<tr>
<th>Step</th>
<th>Perimeter</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

2) Use your calculator to graph a scatter plot and find a line of best fit for the data. Give the equation of the line of best fit:

\[ y = \ldots \]

3) What is the slope of the line of best fit? Explain how you determined this slope.

Slope = _____

Explain: ______________________________________________________

4) What does the slope of the line of best fit represent in terms of this pattern?

5) Extension

What would be the perimeter of the 100th step in the pattern? Explain how you determined this perimeter.

Perimeter = _______

Explain: ______________________________________________________
Worksheet  
Linear Functions in Matchstick Patterns  
Answer Key

Use your matchsticks to create the following stairstep pattern.

1) Determine the perimeter of the pattern for each step.

<table>
<thead>
<tr>
<th>Step</th>
<th>Perimeter</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
</tr>
</tbody>
</table>

2) Use your calculator to graph a scatter plot and find a line of best fit for the data. Give the equation of the line of best fit:

\[ y = 4x \]

3) What is the slope of the line of best fit? Explain how you determined this slope.

Slope = 4

Explain: \textit{The constant difference between the perimeters for each step is 4.}

4) What does the slope of the line of best fit represent in terms of this pattern?

\textit{The slope of the line of best fit (= 4) represents the increase in the perimeter of each shape in successive steps.}

5) Extension

What would be the perimeter of the 100\textsuperscript{th} step in the pattern? Explain how you determined this perimeter.

Perimeter = 400

Explain: \textit{The equation of the line of best fit is } y = 4x. \textit{ Thus, when } x = 100, y = 400.
Development/Procedures:

Lesson 2

Prerequisite Knowledge –– Students should be familiar with
• Linear functions
• Linear regression
• Area

Preassessment – Ask students to define a linear function and to describe some of its key features. Students should know that a linear function is a function that is characterized by a constant rate of change. This is, as the value of one variable changes by a constant amount, the value of the other variable also changes by a constant amount. This concept was developed in Lesson 1.

Launch – Demonstrate to students that they will be making patterns with matchsticks. Show them the first two or three steps in the pattern shown in the Notes handout.

Teacher Facilitation – The teacher should distribute the Notes handout to the students and explain the concepts. The teacher should also circulate around the room to assist students and to monitor their progress.

Student Application – Allow the students to complete the Worksheet in cooperative groups.

Embedded Assessment – Assessment questions are included in Worksheet.

Reteaching/Extension –

- Direct a class discussion in which groups share their answers with the class. The teacher should assist any students who do not understand any of the concepts. A key is included.

- The last question of the Worksheet is an extension question.

- Summary questions are included.

Summary Questions–

Here are some questions to ask the students to summarize the lesson.

1) What is a quadratic function?

A quadratic function is a function in which the differences in successive output values (i.e., the second differences) increase at a constant rate.
2) How is the graph of a quadratic function different from the graph of a linear function?

_The graph of a linear function has a constant slope. The graph of a quadratic function has either increasing slope or decreasing slope._

3) In this lesson, our functions were always increasing. Could a quadratic function be a decreasing function?

_Yes, a portion of the graph of a quadratic function could be decreasing from left to right. For example, consider the graph of y = x^2. The graph is decreasing from left to right until the point (0,0) and then is increasing from left to right. The graphs in this lesson would do the same, except that we only graphed positive x values (because only positive x values made sense in the context of the activity)._
Notes

Quadratic Functions in Matchstick Patterns  Name:________________________

The four shapes pictured below represent the first four shapes in a pattern, where the value of \( n \) represents the step in the pattern.

*Use* your matchsticks to create each of these four shapes.

One may use function notation to represent a given shape property for each step. Here, we will let the function \( f \) represent the total number of small squares in each step. The first step contains one small square, thus one may write that \( f(1) = 1 \). Similarly, the second step contains 4 small squares, so \( f(2) = 4 \). Likewise, \( f(3) = 9 \), \( f(4) = 16 \), and so on.

Use your calculator to enter the input and output values for function \( f \) in a statistics table. To do this, press \[STAT\] and then select EDIT. Put the input values in the column under L1 and the output values in the column under L2. Your display should look like this:

\[
\begin{array}{cccc}
  \text{L1} & \text{L2} & \text{L3} & 1 \\
  1 & 1 & \text{------} \\
  2 & 4 & \text{------} \\
  3 & 9 & \text{------} \\
  4 & 16 & \text{------} \\
\end{array}
\]

\( L1 = \{1, 2, 3, 4\} \).

Note that the differences in successive output values, L2, change at a constant rate. That is, \( 4 - 1 = 3 \), \( 9 - 4 = 5 \), and \( 16 - 9 = 7 \), and so on. Thus the next difference would be 9, the next would be 11, the next would be 13, etc. In other words, the second differences are constant (here, the constant is equal to 2).

Because the differences in successive output values change at a constant rate, this indicates that \( f \) is a quadratic function.
Suppose we wanted to determine the value of \( f(5) \). Because the differences in successive output values are 3, 5, and 7 for \( f(1) \) through \( f(4) \), the next difference would be 9. Thus, \( f(5) \) would be equal to \( f(4) + 9 \), or \( 16 + 9 \), which is 25. So \( f(5) = 25 \).

Next, we will create a scatter plot of this function. To create a scatter plot of the function \( f \), press [WINDOW] and enter the values shown below.

```
WINDOW SETTINGS
Xmin=0
Xmax=5
Xscl=1
Ymin=0
Ymax=20
Yscl=5
Xres=1
```

Now press [2nd] STAT PLOT and turn on Plot 1. Then press [1] and be sure that your calculator screen looks like this:

```
Plot1 Plot2 Plot3
On Off
Type: \( \star \)
Xlist:L1
Ylist:L2
Mark: \( \Box \)
```

First, we will use linear regression to determine the equation of a linear function that best fits these data. To do this, press STAT, then choose CALC, then choose LinReg(ax+b). Press ENTER and then press VARS, and choose Y-VARS and then Y1, so that your screen says LinReg(ax+b) Y1. This will store the regression equation in Y1. Press ENTER.

Then, after you press GRAPH, you should get the following graph:

Note that the line of best fit does not pass through all four points. Thus, a linear model may not be the best fit for the data.
Next, we will use quadratic regression to determine the equation of a quadratic function that best fits these data. To do this, press \textbf{STAT}, then choose \textbf{CALC}, then choose \textbf{QuadReg}.

Press \textbf{ENTER} and then press \textbf{VARS} and choose \textbf{Y-VARS} and then \textbf{Y1}, so that your screen says \textbf{QuadReg Y1}. This will store the regression equation in \textbf{Y1}. Press \textbf{ENTER}.

Then, after you press \textbf{GRAPH}, you should get the following graph:

Note that the quadratic function of best fit passes through all four points. Thus, the function is quadratic.

If you press \textbf{Y=}, you will see that the equation for \textbf{Y1}=X^2. The degree of a function is the value of the greatest exponent in the equation. Thus, the \textbf{degree} of the equation is 2, which is the property of a quadratic function.

In this problem, we only graphed the function for positive values of \(x\), because only positive \(x\) values make sense in terms of the context of the problem. If we were to graph the quadratic function for all values of \(x\), we would see that the graph is decreasing from left to right until the point \((0,0)\), and then increasing from left to right.
Worksheet
Quadratic Functions in Matchstick Patterns       Name:__________________
Use your matchsticks to create the following stairstep pattern.

1) Determine the total number of matchsticks used to create the pattern for each step.

<table>
<thead>
<tr>
<th>Step</th>
<th>Matchsticks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

2) Use your calculator to graph a scatter plot and to determine the line of best fit for the data. Is the function linear? Explain why or why not.

3) Use your calculator to graph a scatter plot and the equation for the quadratic function of best fit for the data. Give the equation of the quadratic function of best fit:

\[ y = \text{________} \]

4) What would be the value of \( f(5) \)?

Value = ______

Explain: ______________________________________________________

5) Extension

How many matchsticks would be used to create the 100th step in the pattern? Explain how you determined this number.

Matchsticks = ______

Explain: ______________________________________________________
Worksheet
Quadratic Functions in Matchstick Patterns       Answer Key

Use your matchsticks to create the following stairstep pattern.

1) Determine the total number of matchsticks used to create the pattern for each step.

<table>
<thead>
<tr>
<th>Step</th>
<th>Matchsticks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>24</td>
</tr>
<tr>
<td>4</td>
<td>40</td>
</tr>
</tbody>
</table>

2) Use your calculator to graph a scatter plot and to determine the line of best fit for the data. Is the function linear? Explain why or why not.  

_The function is not linear because the line of best fit does not pass through all of the data points._

3) Use your calculator to graph a scatter plot and the equation for the quadratic function of best fit for the data. Give the equation of the quadratic function of best fit:

\[ y = 2x^2 + 2x \]

4) What would be the value of f(5)?

Value = 60

Explain: _The differences are 8, 12, and 16, for the first four steps. The next difference would then be 20. Because 40 + 20 = 60, then f(5) = 60._

5) Extension

How many matchsticks would be used to create the 100th step in the pattern? Explain how you determined this number.

Matchsticks = 20200

Explain: _Because the quadratic function of best fit is \( y = 2x^2 + 2x \), then \( f(100) = 2*100^2 + 2*100 = 20200. _
Development/Procedures:

Lesson 3

Prerequisite Knowledge – Students should be familiar with
  • Linear functions
  • Quadratic functions

Preassessment – Ask students if they know the definition of a fractal. Students may not be familiar with this term. A fractal is a figure that results from infinitely many applications of a recursive procedure to a geometric figure.

Launch – Demonstrate to students that they will be making tree fractal patterns with straws. Show them the first two or three steps in the pattern shown in the Notes handout. This should help students visualize what a fractal actually is.

Also ask students to compare the features of linear and quadratic functions. In this lesson, the students will be exploring exponential functions, and will be comparing their properties to those of linear and quadratic functions.

Teacher Facilitation – The teacher should distribute the Notes handout to the students and explain the concepts. The teacher should also circulate around the room to assist students and to monitor their progress.

Student Application – Allow the students to complete the Worksheet in cooperative groups.

Embedded Assessment – Assessment questions are included in Worksheet.

Reteaching/Extension –

  - Direct a class discussion in which groups share their answers with the class. The teacher should assist any students who do not understand any of the concepts. A key is included.

  - The last question of the Worksheet is an extension question.

  - Summary questions are included.

Summary Questions-

Here are some questions to ask the students to summarize the lesson.

1) What is an exponential function?
An exponential function is a function in which the output values increase by a constant factor.

2) How is the graph of an exponential function different from the graph of a linear function?

The graph of a linear function has a constant slope. The graph of an exponential function does not have a constant slope.

3) How is the graph of an exponential function different from the graph of a quadratic function?

A quadratic function is both decreasing and increasing over its domain. An exponential function is either strictly increasing or strictly decreasing over its domain.

4) In this lesson, our exponential functions were always increasing. Could an exponential function be decreasing?

Yes, if the output values change by a factor less than 1. For example, the output values could be 16, 8, 4, 2, …
Exponential Functions in Fractal Patterns

The four shapes pictured below represent the first four shapes in a pattern, where the value of $n$ represents the step in the pattern.

Use your straws to create the following tree fractal pattern. For the first step, use one straw. For the second step, use one straw for the “trunk”, and cut another straw in half for the two branches. Continue similarly for the third and fourth steps.

One may use function notation to represent a given shape property for each step. Here, we will let the function $f$ represent the total number of straw segments in each step. The first step contains one straw segment, thus one may write that $f(1) = 1$. Similarly, the second step contains 3 straw segments, so $f(2) = 3$. Likewise, $f(3) = 7$, $f(4) = 15$, and so on.

Use your calculator to enter the input and output values for function $f$ in a statistics table. To do this, press [STAT] and then select EDIT. Put the input values in the column under L1 and the output values in the column under L2. Your display should look like this:

```
<table>
<thead>
<tr>
<th>L1</th>
<th>L2</th>
<th>L3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td></td>
</tr>
</tbody>
</table>
```

$L1 = \{1, 2, 3, 4\}$

Note that the differences in successive output values, L2, double for each step. That is, $3 - 1 = 2$, $7 - 3 = 4$, and $15 - 7 = 8$, and so on. Thus the next difference would be 16, the next would be 32, the next would be 64, etc.
Because the differences in successive output values are multiplied by a constant factor (here each successive difference is doubled, although it could be tripled, quadrupled, etc.), this indicates that $f$ is **exponential function**.

Suppose we wanted to determine the value of $f(5)$. Because the differences in successive output values are 2, 4, and 8 for $f(1)$ through $f(4)$, the next difference would be 16. Thus, $f(5)$ would be equal to $f(4) + 16$, or $15 + 16$, which is 31. So $f(5) = 31$.

Next, we will create a scatter plot of this function. To do this, press **WINDOW** and enter the values shown below:

\[
\begin{align*}
X_{\text{min}} &= 0 \\
X_{\text{max}} &= 5 \\
X_{\text{scl}} &= 1 \\
Y_{\text{min}} &= 0 \\
Y_{\text{max}} &= 20 \\
Y_{\text{scl}} &= 5 \\
X_{\text{res}} &= 1
\end{align*}
\]

Now press **2nd** [STAT PLOT] and turn on Plot 1. Then press **1** and be sure that your calculator screen looks like this:

\[
\begin{align*}
\text{Plot1} &: \text{On} \\
\text{Type} &: \text{L} \\
X_{\text{list}} &: \text{L1} \\
Y_{\text{list}} &: \text{L2} \\
\text{Mark} &: \text{ } \text{ }
\end{align*}
\]

Next, we will use exponential regression to determine the equation of an exponential function that best fits this data set. To do this, press **STAT**, then choose **CALC**, then choose **ExpReg**. Press **ENTER** and then select **VARS**, and choose **Y-VARS** and then **Y1**, so that your screen says **ExpReg Y1**. This will store the regression equation in **Y1**. Press **ENTER**.

Then, after you press **GRAPH**, you should get the following graph, which represents the regression equation $y = 0.4472 \times 2.453^x$:

![Graph of exponential function](image-url)
The function appears to be a close fit, but it does not pass through each data point. The reason for this is that in exponential regression, the function is forced to be of the form \( y = a \times b^x \). Our data would have fit this model if the L2 values had been 2, 4, 6, 8, … instead of 1, 3, 7, 15 … (that is, we needed to subtract 1 from each L2 value). In that case, because the first term would be 2, and then each successive term would double, the exponential equation \( y = 2^x \) would fit the data.

We can find an exponential model that accounts for this shift of 1 unit by using the Interactive Graphing Application. To do this, press [APPs] then choose “Interact”, then choose CONTINUE. Hit \[ \text{Y=} \] and store the following regression equation in Y1: \( y = a \times b^x + c \). Your screen should look like this:

```
  Plot1  Plot2  Plot3
MY1=A*B^(X)+C
MY2=
MY3=
MY4=
MY5=
MY6=
MY7=
```

Because we noted that we wanted to subtract 1 from all of our L2 values, we can see that \( C \) should be equal to -1. Even if we were unsure of this, Interactive Graphing would allow us to test different values. We are unsure of \( A \) and \( B \). Now, press [WINDOW] and select SETTINGS. Set the values of \( A \), \( B \), \( C \), and STEP as follows:

```
WINDOW SETTINGS
\[ \text{H=0} \]
\[ \text{B=0} \]
\[ \text{C=-1} \]
\[ \text{Step=.5} \]
```

Then, after you press [GRAPH], you should see a scatter plot of your data along with an exponential function that does not match the data points. This indicates that \( A = 0 \) and \( B = 0 \) are not the correct values. Thus, you will need to adjust the values of \( A \) and \( B \) by selecting these values separately and pressing the left or right arrow keys to decrease or increase \( A \) or \( B \) by the specified step value of 1.

You will find that the following values of \( A \), \( B \), and \( C \) give a perfect fit:
Note that the exponential function of best fit passes through all four points. Thus, the function is exponential. Because $A = 1$, $B = 2$, and $C = -1$, the equation of best fit is $y = 2^x - 1$.

You may verify that the exponential function of best fit passes through all four data points by examining the table on your calculator. To do this, first set up the table by pressing $2^{\text{nd}}$ TBLSET and setting up the table with the following values:

<table>
<thead>
<tr>
<th>TABLE SETUP</th>
</tr>
</thead>
<tbody>
<tr>
<td>TblStart = 1</td>
</tr>
<tr>
<td>$\Delta$Tbl = 1</td>
</tr>
<tr>
<td>Indpt: Auto Ask</td>
</tr>
<tr>
<td>Depend: Auto Ask</td>
</tr>
</tbody>
</table>

Now, press $2^{\text{nd}}$ TABLE and you will see:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.0</td>
</tr>
<tr>
<td>2</td>
<td>15.0</td>
</tr>
<tr>
<td>3</td>
<td>31.0</td>
</tr>
<tr>
<td>4</td>
<td>63.0</td>
</tr>
</tbody>
</table>

Note that the values in the $Y_1$ column match the values that were entered into statistics list L2, so the exponential function is a perfect fit to the data set.
Worksheet
Exponential Functions in Fractal Patterns

Use your straws to create the following tree fractal pattern.

1) Determine the number of **new** straw segments in the pattern for each step.

<table>
<thead>
<tr>
<th>Step</th>
<th>New segments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

2) Use your calculator to graph a scatter plot and find an exponential function of best fit for the data. Use either Interactive Graphing (set Y1 = AB^X) or Exponential Regression. Give the equation of the exponential function of best fit:

\[ y = \ldots \]

3) Based on the data, why do you think that an exponential model is the most appropriate model for this situation?

4) Extension

How many new segments would be in the 100\textsuperscript{th} step in the pattern? Explain how you determined this number.

New segments = \ldots

Explain: \ldots
Worksheet
Exponential Functions in Fractal Patterns

Use your straws to create the following tree fractal pattern.

1) Determine the number of new straw segments in the pattern for each step.

<table>
<thead>
<tr>
<th>Step</th>
<th>New segments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>

2) Use your calculator to graph a scatter plot and find an exponential function of best fit for the data. Use either Interactive Graphing (set \( Y1 = AB^X(X) \)) or Exponential Regression. Give the equation of exponential function of best fit:

\[ y = 0.5 \times 2^x \]

3) Based on the data, why do you think that an exponential model is the most appropriate model for this situation?

Because the number of new segments in each successive step doubles. Also, the successive differences double (2 – 1 = 1, 4 – 2 = 2, 8 – 4 = 4, etc.)

4) Extension

How many new segments would be in the 10th step in the pattern? Explain how you determined this number.

New segments = 512

Explain: \( f(10) = 0.5 \times 2^{10} = 1024 \)
Summative Assessment

Students will be assessed on three types of functions – linear, quadratic, and exponential.

1. Linear function

The table below shows the percentage of diesel trucks sold in the U.S. from 1970 to 2000.

Percentage of Total Diesel Trucks Sold in the U.S

<table>
<thead>
<tr>
<th>Year</th>
<th>Percent of Total Sold</th>
</tr>
</thead>
<tbody>
<tr>
<td>1970</td>
<td>94</td>
</tr>
<tr>
<td>1975</td>
<td>82</td>
</tr>
<tr>
<td>1980</td>
<td>64</td>
</tr>
<tr>
<td>1985</td>
<td>56</td>
</tr>
<tr>
<td>1990</td>
<td>32</td>
</tr>
<tr>
<td>1995</td>
<td>25</td>
</tr>
<tr>
<td>2000</td>
<td>9</td>
</tr>
</tbody>
</table>

a) Make a scatter plot of the data below.

Scatter Plot:

b) Use a graphing calculator to find the equation of the line of the best fit. Let 1970 = 70, 1975 = 75, etc.

Equation of Line of Best Fit: ______________
2. Quadratic function

The table below shows the attendance at men’s college basketball games from 1977-1983

**Attendance at Men’s College Basketball Games**

<table>
<thead>
<tr>
<th>Year</th>
<th>Attendance (thousands)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1977</td>
<td>2157</td>
</tr>
<tr>
<td>1978</td>
<td>2326</td>
</tr>
<tr>
<td>1979</td>
<td>2501</td>
</tr>
<tr>
<td>1980</td>
<td>2778</td>
</tr>
<tr>
<td>1981</td>
<td>3014</td>
</tr>
<tr>
<td>1982</td>
<td>3398</td>
</tr>
<tr>
<td>1983</td>
<td>4192</td>
</tr>
</tbody>
</table>

a) Find the quadratic model for the attendance at men’s basketball games from 1977 - 1983. Hint: use quadratic regression where \( x = 0 \) represents 1970.

b) Use your equation to predict the attendance at men’s college basketball games (in thousands) in 1987.
3. Exponential function

Bacteria in a laboratory culture can double in number every 20 min. Suppose a culture starts with 25 cells.

   a) Copy, complete, and extend the table to find when there will be more than 1,000 bacteria cells.

<table>
<thead>
<tr>
<th>Time</th>
<th>Number of 20 min time periods</th>
<th>Pattern</th>
<th>Number of cells</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial</td>
<td>0</td>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>20 min</td>
<td>1</td>
<td>25*2</td>
<td>25*2 = 50</td>
</tr>
<tr>
<td>40 min</td>
<td>2</td>
<td>25<em>2</em>2</td>
<td>25*4 = 100</td>
</tr>
</tbody>
</table>

   b) How long will it take to reach 1,000 bacteria cells?

   c) Write a general rule for the \( n \)th 20-minute time period.

   d) What will be the total number of bacteria after 3 hours (nine 20 minute time periods)?
4. Quadratic Function

Consider the following data from the Rockingham County, Virginia school district, which relates teacher salary to years of experience.

<table>
<thead>
<tr>
<th>Years of Experience</th>
<th>10 Months/200 Days</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$33,200</td>
</tr>
<tr>
<td>1</td>
<td>$33,200</td>
</tr>
<tr>
<td>2</td>
<td>$33,200</td>
</tr>
<tr>
<td>3</td>
<td>$33,700</td>
</tr>
<tr>
<td>4</td>
<td>$33,700</td>
</tr>
<tr>
<td>5</td>
<td>$33,700</td>
</tr>
<tr>
<td>6</td>
<td>$33,700</td>
</tr>
<tr>
<td>7</td>
<td>$34,200</td>
</tr>
<tr>
<td>8</td>
<td>$34,200</td>
</tr>
<tr>
<td>9</td>
<td>$34,400</td>
</tr>
<tr>
<td>10</td>
<td>$34,700</td>
</tr>
<tr>
<td>11</td>
<td>$35,300</td>
</tr>
<tr>
<td>12</td>
<td>$35,700</td>
</tr>
<tr>
<td>13</td>
<td>$36,200</td>
</tr>
<tr>
<td>14</td>
<td>$36,700</td>
</tr>
<tr>
<td>15</td>
<td>$37,500</td>
</tr>
<tr>
<td>16</td>
<td>$38,250</td>
</tr>
<tr>
<td>17</td>
<td>$38,750</td>
</tr>
<tr>
<td>18</td>
<td>$39,250</td>
</tr>
<tr>
<td>19</td>
<td>$40,000</td>
</tr>
<tr>
<td>20</td>
<td>$41,000</td>
</tr>
<tr>
<td>21</td>
<td>$42,000</td>
</tr>
<tr>
<td>22</td>
<td>$42,600</td>
</tr>
<tr>
<td>23</td>
<td>$43,100</td>
</tr>
<tr>
<td>24</td>
<td>$43,700</td>
</tr>
<tr>
<td>25</td>
<td>$44,400</td>
</tr>
<tr>
<td>26</td>
<td>$45,400</td>
</tr>
<tr>
<td>27</td>
<td>$45,900</td>
</tr>
<tr>
<td>28 &amp; over</td>
<td>$47,850</td>
</tr>
</tbody>
</table>

Data source: Rockingham County, VA Public Schools

a) Find a quadratic regression equation for the data.

b) Do you believe that a quadratic model is appropriate for this data set? Justify your answer.
5. Exponential Function

Examine the table below:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>800</td>
<td>400</td>
<td>200</td>
<td>100</td>
<td>50</td>
<td>25</td>
</tr>
</tbody>
</table>

a) Which type of function (linear, quadratic, exponential) best fits the data set? Justify your answer.

b) Find an equation for the curve of best fit.

c) Determine the $y$-value when $x = 40$. Explain how you determined your answer.
Summative Assessment Answer Key

Students will be assessed on three types of functions – linear, quadratic, and exponential.

1. Linear function

The table below shows the percentage of diesel trucks sold in the U.S. from 1970 to 2000.

**Percentage of Total Diesel Trucks Sold in the U.S**

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<td>25</td>
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<tr>
<td>2000</td>
<td>9</td>
</tr>
</tbody>
</table>

a) Make a scatter plot of the data below.
Scatter Plot:

```
\[ \text{Scatter Plot:} \]
```

b) Use a graphing calculator to find the equation of the line of the best fit. Let 1970 = 70, 1975 = 75, etc.

Equation of Line of Best Fit: \( y = -2.864x + 295.179 \)
2. Quadratic function

The table below shows the attendance at men’s college basketball games from 1977-1983

<table>
<thead>
<tr>
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<td>1982</td>
<td>3398</td>
</tr>
<tr>
<td>1983</td>
<td>4192</td>
</tr>
</tbody>
</table>


\[ y = 48.667x^2 - 660.405x + 4452.143 \]

b) Use your equation to predict the attendance at men’s college basketball games (in thousands) in 1987.

7,290
3. Exponential function

Bacteria in a laboratory culture can double in number every 20 min. Suppose a culture starts with 25 cells.

a) Copy, complete, and extend the table to find when there will be more than 1,000 bacteria cells.

<table>
<thead>
<tr>
<th>Time</th>
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</tr>
<tr>
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<td>25 * 2</td>
<td>25 * 2 = 50</td>
</tr>
<tr>
<td>40 min</td>
<td>2</td>
<td>25 * 2 * 2</td>
<td>25 * 4 = 100</td>
</tr>
<tr>
<td>60 min</td>
<td>3</td>
<td>25 * 2 * 2 * 2</td>
<td>25 * 8 = 200</td>
</tr>
<tr>
<td>80 min</td>
<td>4</td>
<td>25 * 2 * 2 * 2 * 2</td>
<td>25 * 16 = 400</td>
</tr>
</tbody>
</table>

b) How long will it take to reach 1,000 bacteria cells?

106.438 minutes (or 5.3219 twenty-minute time periods)

c) Write a general rule for the nth 20-minute time period.

\[ c(n) = 25 \cdot 2^n \]

d) What will be the total number of bacteria after 3 hours (nine 20 minute time periods)?

12800
4. Quadratic Function

Consider the following data from the Rockingham County, Virginia school district, which relates teacher salary to years of experience.

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<thead>
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</tr>
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<td>4</td>
<td>$33,700</td>
</tr>
<tr>
<td>5</td>
<td>$33,700</td>
</tr>
<tr>
<td>6</td>
<td>$33,700</td>
</tr>
<tr>
<td>7</td>
<td>$34,200</td>
</tr>
<tr>
<td>8</td>
<td>$34,200</td>
</tr>
<tr>
<td>9</td>
<td>$34,400</td>
</tr>
<tr>
<td>10</td>
<td>$34,700</td>
</tr>
<tr>
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<tr>
<td>14</td>
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</tr>
<tr>
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<tr>
<td>18</td>
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</tr>
<tr>
<td>19</td>
<td>$40,000</td>
</tr>
<tr>
<td>20</td>
<td>$41,000</td>
</tr>
<tr>
<td>21</td>
<td>$42,000</td>
</tr>
<tr>
<td>22</td>
<td>$42,600</td>
</tr>
<tr>
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<td>$43,100</td>
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<td>26</td>
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<tr>
<td>27</td>
<td>$45,900</td>
</tr>
<tr>
<td>28 &amp; over</td>
<td>$47,850</td>
</tr>
</tbody>
</table>

Data source: Rockingham County, VA Public Schools

c) Find a quadratic regression equation for the data.

\[ y = 0.0177x^2 + 0.2176x + 33.1162 \]
d) Do you believe that a quadratic model is appropriate for this data set? Justify your answer.

_The model appears to fit very well. However, it may not be accurate past 28 years because salaries level out after then, whereas the quadratic equation continues to increase in value._

5. Exponential Function

Examine the table below:

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>800</td>
<td>400</td>
<td>200</td>
<td>100</td>
<td>50</td>
<td>25</td>
</tr>
</tbody>
</table>

a) Which type of function (linear, quadratic, exponential) best fits the data set? Justify your answer.

_Exponential. Each successive y-value is cut in half. Also, successive differences are cut in half each time: 800 – 400 = 400, 400 – 200 = 200, 200 – 100 = 100, etc._

b) Find an equation for the curve of best fit.

\[ y = 800 \times 0.8706^x \]

c) Determine the y-value when x = 40. Explain how you determined your answer.

_The value of y is halved every time that the value of x increases by 5._

_Thus, when x = 30, y = 12.5, when x = 35, y = 6.25, and when x = 40, y = 3.125._
Bibliography


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