Title: Making Money and Spreading the Flu! *Using Real-Life Models to Understand Properties of Exponential Growth*

Brief Overview:

Students will use exponential functions to model exponential growth in several real-life situations: spread of flu in a school, bacteria growth and the continuously compounded interest.

NCTM Content Standards:

Number and Operations

The students will understand the relationship between pairs of numbers, ways of representing numbers and patterns in numbers.
The students will compute fluently and make reasonable predictions.

Algebra

The students will understand Patterns, Relations, and Functions.
The students will use mathematical models to represent a situation and analyze changes in situation.
The students will draw reasonable conclusions about a situation being modeled.

Data Analysis and Probability

The students will formulate questions, and will collect, organize, and display data to answer them.
The students will find the equation that model the data using technology.
The students will make and evaluate inferences and predictions based on the data.

NCTM Process Strands:

Problem Solving

The students will build new mathematical knowledge through problem solving.

Communication

The students will organize and consolidate their mathematical thinking through communication.

Connections

The students will understand how mathematical ideas interconnect and build on one another to produce a coherent whole.

Representation

The students will make and use representations to organize, record, model and communicate mathematical ideas.

Grade/Level:

Algebra II, grade 10/11

Duration/Length:

Three 45-minute classes for Lessons 1, 2, and 3.

Prerequisite Knowledge:

Students should have working knowledge of the following skills:

- Identify dependent and independent variables for a data set
- Recognize patterns in plotted data
- Describe linear and quadratic functions
- Apply compound interest formula

Student Outcomes:

Students will be able to:

- Identify exponential growth as a model of real-life situations
- Construct and make inferences from graphs of exponential functions

Materials and Resources:

- Colored Paper
- Worksheets
- Coordinate plane boards
- Markers
- Monopoly Play Money
- Assessment Sheets
- TI-83+ graphing calculator

Development/Procedures:

LESSON 1: Ideas that Lead to the Concept of Exponential Growth

Preassessment (10-15min)

Review functions previously studied such as linear and quadratic, by having students complete an exercise to recall the algebraic and geometric characteristics of these functions:

- Give an example of a linear relationship. What is the slope-intercept form for the equation of a line?
- Write the standard form of the quadratic equation. Graph an example and describe key features of the graph.

Launch

Students will engage in a hands-on activity to investigate the relationship between the number of folds on a piece of paper and the number of rectangles produced. They will fold rectangles successively, producing "exponentially more" rectangles with each fold. Distribute lesson 1 activity worksheet and sheets of colored paper to each student. Read and discuss the directions to the students putting emphasis on the questions that need to be answered. Have the students work in cooperative groups of three.

Pose the problem to the groups. Demonstrate by folding a sheet of paper in 3 equal parts. Have them count the number of rectangles formed after folding the colored paper. Assist the groups to come up with at least three (3) entries on the table.

Teacher Facilitation

- Organize students into groups of three to participate in the cooperative group activity.
- Circulate around the room to offer assistance and make sure students are on task.

Student Application (20-25min)

Have students continue folding in thirds and recording the number of rectangles formed until they can no longer make another fold. Ask groups to describe the pattern that is emerging from the data. Distribute the activity worksheet. Have students do the activity and record the data. Encourage students to communicate their observations with others. Allow the students to grow into knowledge of the concept on their own. Actively engage the students in problem solving.

Embedded Assessment

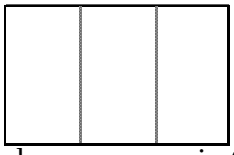
Check students' work to make sure that all students are working.

Reteaching/Extension (10-15min)

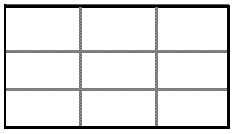
- Bring class together to share observations on the relationship between folding the paper and the number of rectangles formed.
- Focus on articulating a clear statement of an “exponential relationship.”
- Have students graph their points (teacher produces graph on board/overhead).
- Teacher-directed class discussions on key features of the graph of the exponential function.
- Have students make comparisons between the exponential graph and linear/quadratic models.
- Distribute assessment.

LESSON 1 Activity Worksheet: Paper Folding

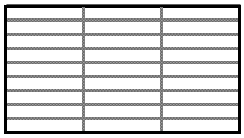
Directions:



1. Fold a sheet of paper into 3 equal parts.
   Count the number of rectangles formed. Record your answer in the table.



2. Fold the paper again into 3 equal parts.
   Count the number of rectangles formed. Record your answer in the table.



3. Continue folding the paper into 3 equal parts, counting and recording the number of rectangles formed until you can no longer make another fold.

Fold Stage	**Number of Rectangles Formed**
0	1
1	
2	
3	
4	
5	
:	
:	
:	
n	

4. Examine the data gathered in the table. Did you see any pattern in the number of rectangles formed as the fold stage increases? Describe how the number of rectangles formed changes as the fold stage increases.

   _____

   _____

   _____

5. Write a rule to find the number of rectangles formed if the paper was folded ***n*** times

   _____

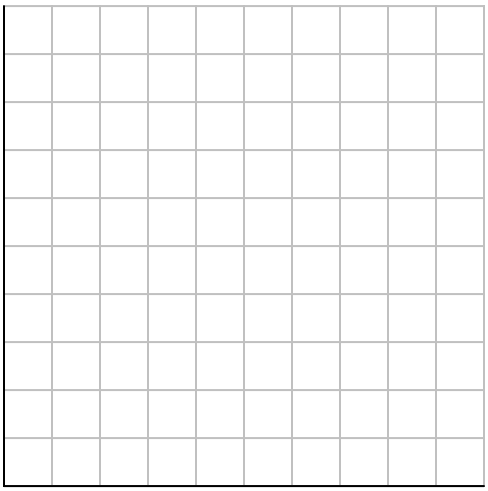
6. Predict the number of rectangles after 10 folds.

_____

7. If your paper now has 6561 rectangles, how many folds would you have made?

_____

8. Make a sketch of the scatter plot on the grid below. Describe the graph.



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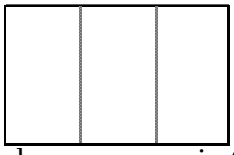
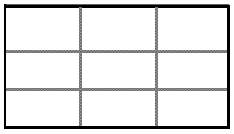
LESSON 1 Activity Worksheet: Paper Folding (ANSWER KEY)

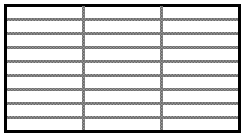
Directions:



1. Fold a sheet of paper into 3 equal parts.
   Count the number of rectangles formed. Record your answer in the table.



2. Fold the paper again into 3 equal parts.
   Count the number of rectangles formed. Record your answer in the table.



3. Continue folding the paper into 3 equal parts, counting and recording the number of rectangles formed until you can no longer make another fold.

Fold Stage	**Number of Rectangles Formed**
0	1
1	***3***
2	***9***
3	***27***
4	***81***
5	***243***
:	***:***
:	***:***
:	***:***
n	3^n

4. Examine the data gathered in the table. Did you see any pattern in the number of rectangles formed as the fold stage increases? Describe how the number of rectangles formed changes as the fold stage increases.

Fold Number	**Number of Rectangles Formed**	**Folding Process**
0	1	
1	**3**	$3^1 = 3$
2	**9**	$3^2 = 3 \times 3$
3	***27***	$3^3 = 3 \times 3 \times 3$
4	***81***	$3^4 = 3 \times 3 \times 3 \times 3$
5	***243***	$3^5 = 3 \times 3 \times 3 \times 3 \times 3$

<u>Sample Solution</u>:
More rectangles are formed as you keep on folding the paper into 3 equal parts. However, the increase is not constant. Therefore, this can't be linear. The values for the number of rectangles formed were all multiples of 3. The number of factors was the same as the number of times the paper was folded, and a pattern of exponents existed.

5. Write a rule to find the number of rectangles formed from the number of times the paper was folded.

Let r – number of rectangles
f - fold stage (number of times you fold a paper in three equal parts)

$$r = 3^{f}$$

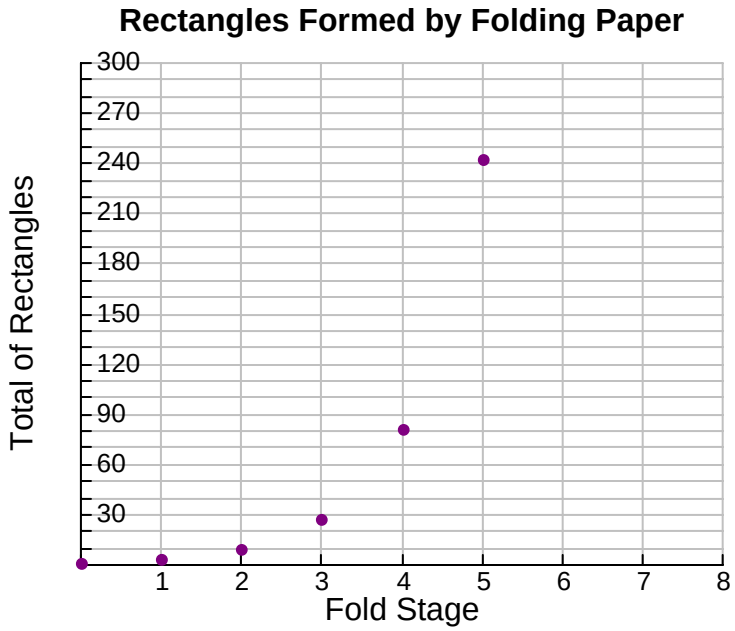
6. Predict the number of rectangles after 10 folds.

59049

7. If your paper now has 6561 rectangles, how many folds would you have made already?

8

8. Make a sketch of the scatter plot on the grid below. Describe the graph.



Assessment

NAME_____DATE_____

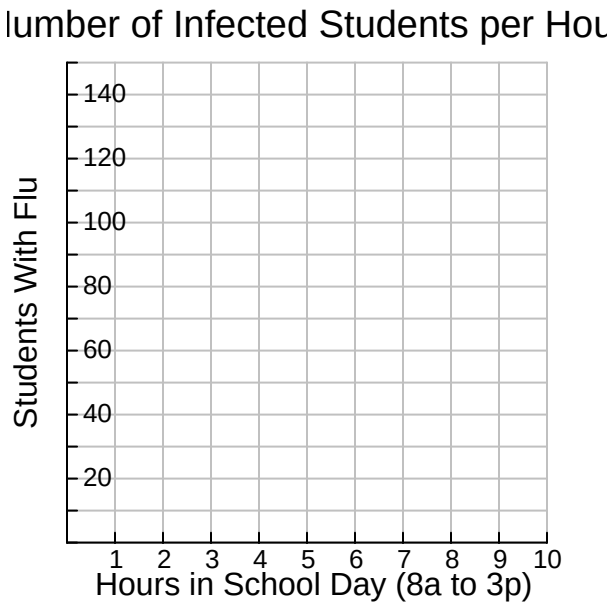
Modeling the Spread of Flu at a High School

1. Mike is sick with the flu but he still comes to school on Monday. He arrives at 8am and by 9am *(Hour 1)*, Mike has already infected two of his friends, Jose and Lisa. By 10am *(Hour 2)*, Jose has infected two of his friends, Bill and Ted, and Lisa has infected two of her friends, Jenny and Kiara.

(a) If each person with the flu infects two other people in each hour, how many students are infected by 3pm *(Hour 7)*? Complete the Table and graph the points in the grid below.

Table: *Number of Students Infected With Flu By Hour*

Hour 0	Hour 1	Hour 2	Hour 3	Hour 4	Hour 5	Hour 6	Hour 7
1	2	4					



(b) If the school has 1,000 students enrolled, what percentage of the student body has the flu by 3pm?

Cicada Population

2. Assuming the population of cicada doubles every week. Answer the following questions to complete the table below:

Week	**Number of Cicada**
0	3000
1	
2	
3	
4	
5	
:	
:	
:	
x	

a) If we begin with 3,000 cicadas, how many will we have after 2 weeks?

_____

b) If we begin with 3,000 cicadas, how many will we have after 5 weeks?

_____

c) If we begin with 3,000 cicadas, how many will we have after x weeks?

_____

d) If we begin with 3,000 cicadas, after how many weeks will we have 24,000 cicadas?

_____

e) If we begin with 3,000 cicadas, after how many weeks will we have 20,000 cicadas?

_____

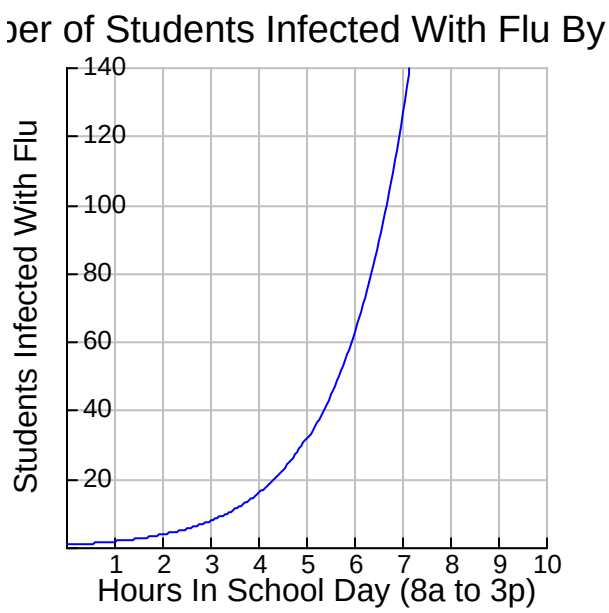
Assessment – Answer Key

1. Mike is sick with the flu but he still comes to school on Monday. He arrives at 8am and by 9am *(Hour 1)*, Mike has already infected two of his friends, Jose and Lisa. By 10am *(Hour 2)*, Jose has infected two of his friends, Bill and Ted, and Lisa has infected two of her friends, Jenny and Kiara.

(a) If each person with the flu infects two other people in each hour, how many students are infected by 3pm *(Hour 7)*? Complete the Table and graph the points in the grid below.

Table: *Number of Students Infected With Flu By Hour*

Hour 0	Hour 1	Hour 2	Hour 3	Hour 4	Hour 5	Hour 6	Hour 7
1	2	4	**8**	**16**	**32**	**64**	**128**



(b) If the school has 1,000 students enrolled, what percentage of the student body has the flu by 3pm? ***25.5%***

Cicada Population

2. Assuming the population of cicada doubles every week. Answer the following questions to complete the table below:

Week	**Number of Cicada**
0	3000
1	***6000***
2	***12000***
3	***24000***
4	***48000***
5	***96000***
:	
:	
:	
x	$3000 \cdot 2^x$

a) If we begin with 3,000 cicadas, how many will we have after 2 weeks?

12000

b) If we begin with 3,000 cicadas, how many will we have after 5 weeks?

96000

c) If we begin with 3,000 cicadas, how many will we have after ***x*** weeks?

Let n = number of cicadas
x = no. of weeks

$n = 3000 \cdot (2^x)$

d) If we begin with 3,000 cicadas, after how many weeks will we have 24,000 cicadas?

3 weeks

e) If we begin with 3,000 cicadas, after how many weeks will we have 20,000 cicadas?

2.737 weeks

LESSON 2: Continuously Compounded Interest and e^x

Preassessment

The students should have knowledge of the compounded interest formula:

$$A = P(1+\frac{r}{n})^{nt}.$$

Launch

Students will:

- Describe properties of an exponential growth function
- Plot data points to construct a graph of an exponential function

Teacher Facilitation

Explain ground rules of *"Increasing Interest: How To Create Wealth (And Have Fun)."* Assign student volunteer as Banker and form four groups of three to four students each. Distribute worksheet.

Student Application
Cooperative Group Activity

<u>Increasing Interest! How to Create Wealth (and have Fun!)</u>

- One student will volunteer/ be assigned as Banker for the class. Students will form four Groups (A, B, C, D) with four to five students per group. **Group A** will have the following amounts: $100, $200, $300, $400; **Group B** will have the following amounts: $250, $500, $750, $1000; **Group C** will have the following amounts: $500, $1000, $1500, $2000; **Group D** will have the following amounts: $1000, $2000, $3000, $4000.

- The Banker will randomly choose group assignments. Within each group, members will choose their amounts (e.g., one person receives $100; another receives $200 in **Group A** and so on).

- Students will receive their share in Monopoly cash. The Banker will set the interest rate at any amount between 5%-10%. Students will calculate interest during *t=1 year, t=5 years, and t=10 years*. *Year 1* will be the first 2 minutes, *Year 5* will be for 10 minutes, and *Year 10* will be for 20 minutes.

- For their individual amount, the student is responsible for applying the Compound Interest Formula and calculating their savings *Monthly, Weekly, and Daily.*

Students work in groups of three to four to collectively calculate total savings after interest has been compounded. Students will calculate interest Yearly, Monthly, Weekly, and Daily. The Banker will pay interest dividends after each time period. Students will then plot data on Class Data Chart.

Embedded Assessment

Teacher will check that students are correctly applying the compound interest formula.

Reteaching/Extension

The teacher will focus students on their developing concept of Exponential Growth by asking: *What trend do we notice when we increase the number of times interest is compounded? Can we increase this compounding? By how much?*

At this stage, the teacher should motivate the concept of *continuously* compounding interest. The teacher should ask: *As n in the formula increases, as we compound interest more and more, what function are we beginning to see approximated? (Hint: think about the previous lesson!)*

The teacher then introduces the continuously compounded interest formula: $A = Pe^{rt}$. The teacher guides students to graph $y=e^x$ on the TI-83. The teacher asks students to describe the relationship between the pattern they see in number of times interest is compounded and $y=e^x$.

LESSON 2: Worksheet For *Increasing Interest!*

Group A

Daily				
Weekly				
Monthly				
Yearly				
Continuous				

Group B

Daily				
Weekly				
Monthly				
Yearly				
Continuous				

Group C

Daily				
Weekly				
Monthly				
Yearly				
Continuous				

Group D

Daily				
Weekly				
Monthly				
Yearly				
Continuous				

LESSON 3: Applications of Exponential Growth – Extensions and Connections

Preassessment (5-10min)

The teacher will review concepts from Lesson 2 and Lesson 1. Direct students to brainstorm analogous situations to "Folding Rectangles" and "Spreading Flu." The Goal of Lesson 3 is connect the applications of the exponential function to its definition and properties. The purpose is to compare the gathered data by graphing.

Launch (10-15min)

Students compare transformations of $f(x)=e^x$ using Interactive Graphing. Distribute worksheet.

Teacher Facilitation

The teacher will guide students to correctly transform the exponential function and connect to previous applications and examples. Distribute **Summative Assessment**. The teacher will ask students to go to their assigned groups. Each group will be given a coordinate plane board, markers, and a worksheet.

Student Application

Students work in pairs to complete Assessment. Using the data which the students gathered and recorded on the worksheet, they will enter the data into their calculators, graph, and describe its behavior.

Embedded Assessment

Teacher will frequently check students' progress.
Teacher will circulate around the room to provide assistance/refocus questions as needed.

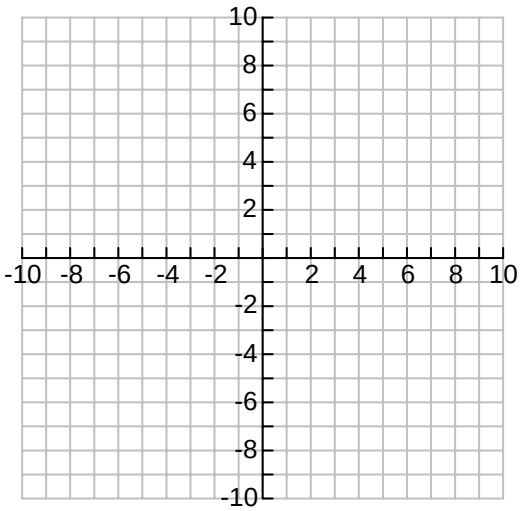
Reteaching/Extension

- The teacher will create a small group for lesson review.
- The teacher will review any needed material/information.
- The students will write and explain the following:
  "Which compound model yields the highest return?"

EXPONENTIAL TRANSFORMATION WORKSHEET

Today your group will explore transformations of exponential functions of the form $y = e^{ax}$, *where* $0 < a < 1$. You will use the application program *Interactive Graphing* on the TI-83+. Your group will discover how different values of *a* affect the shape of the graph.

Graph the parent exponential function $y = e^{x}$ on the grid below and complete the table of values.



x	y
-6	
-4	
0	
4	
6	

How does the value of *a* affect the graph of $y = e^{ax}$?

1. Open the Interactive application under the APPS key on your calculator. Type in the equation $y = e^{ax}$ into Y=. Set a standard window to match the graph above. Under the window screen press the Up Arrow to Adjust Settings. Set $a=1$. What equation are you graphing? Examine the table to verify your answer.

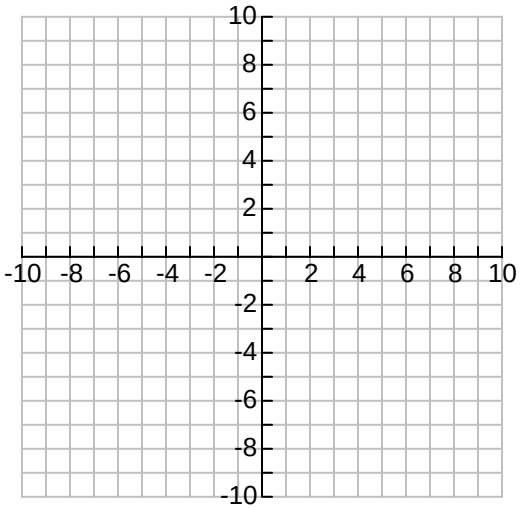
2. Now change the value of *a*. You may do this on the graph screen by typing in a new value for *a* or using the left arrows. Try the following values of *a* and complete the table:

Value of *a*	**Equation**	**How graph is modified**
$a=7$	$y=e^{7x}$	
$a=3$		
$a=0.5$		
$a=0.2$		

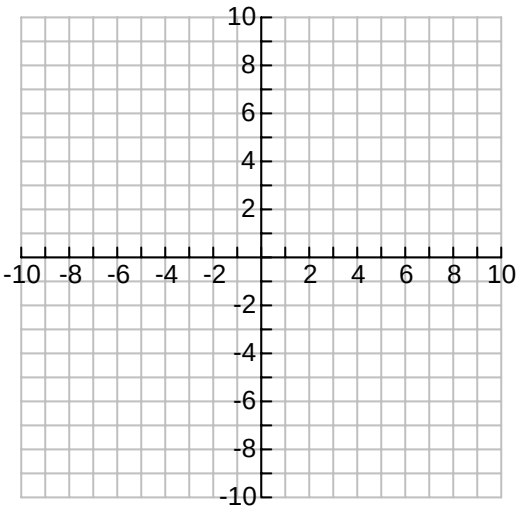
3. Now write a general rule explaining how the value of *a* affects the equation $y=e^{ax}$.

4. Now sketch an example of a graph where the value of *a* is:

(a) Greater than 1



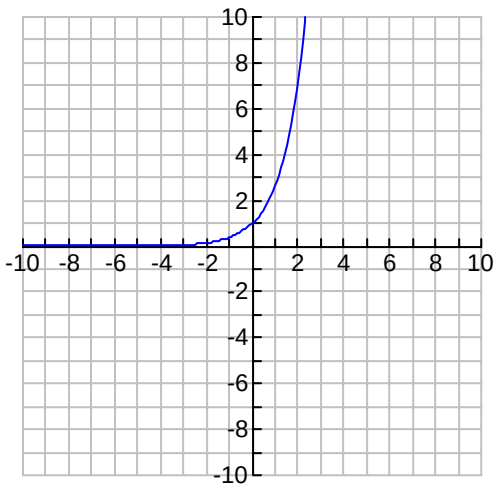
(b) A fraction between 0 and 1



EXPONENTIAL TRANSFORMATION WORKSHEET – ANSWER KEY

Today your group will explore transformation of exponential functions of the form $y = e^{ax}$, *where* $0 < a < 1$. You will use the application program *Interactive Graphing* on the TI-83+. Your group will discover how different values of *a* affect the shape of the graph.

Graph the parent exponential function $y = e^{x}$ on the grid below and complete the table of values.



x	y
-5	**0.007**
-3	**0.005**
0	**1.000**
3	**20.086**
5	**148.413**

How does the value of *a* affect the graph of $y = e^{ax}$?

5. Open the Interact application under the APPS key on your calculator. Type in the equation $y = e^{ax}$ into Y=. Set a standard window to match the graph above. Under the window screen press the Up Arrow to Adjust Settings. Set $a=1$. What equation are you graphing? Examine the table to verify your answer.

6. Now change the value of *a*. You may do this on the graph screen by typing in a new value for *a* or using the left arrows. Try the following values of *a* and complete the table:

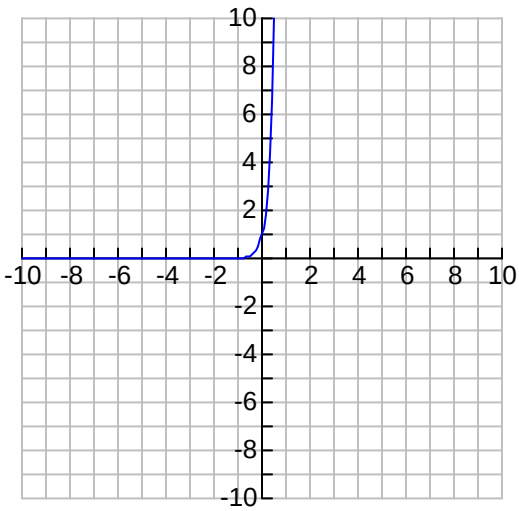
Value of *a*	Equation	How graph is modified
$a=7$	$y=e^{7x}$	**Contracts towards y-axis**
$a=3$	$\boldsymbol{y=e^{3x}}$	**Contracts towards y-axis**
$a=0.5$	$\boldsymbol{y=e^{0.5x}}$	**Expands towards x-axis**
$a=0.2$	$\boldsymbol{y=e^{0.2x}}$	**Expands towards x-axis**

7. Now write a general rule explaining how the value of *a* affects the equation $y=e^{ax}$.

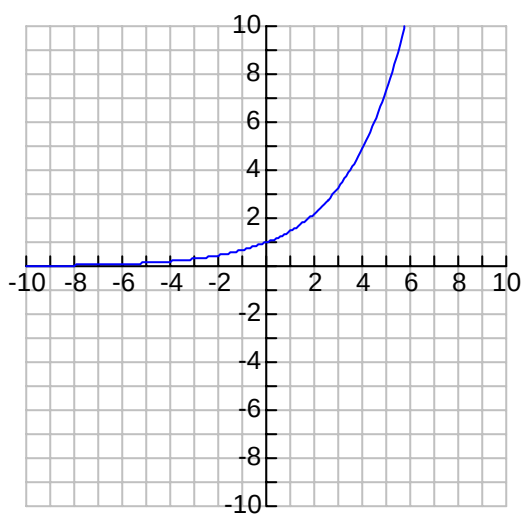
If* a > 1, *the graph contracts in towards y-axis ("increases faster").
If* 0 < a < 1, *the graph expands out towards the x-axis ("increases slower").

8. Now sketch an example of a graph where the value of *a* is:

(a) greater than 1 $\boldsymbol{y = e^{5x}}$



(b) a fraction between 0 and 1 $\boldsymbol{y = e^{0.4x}}$



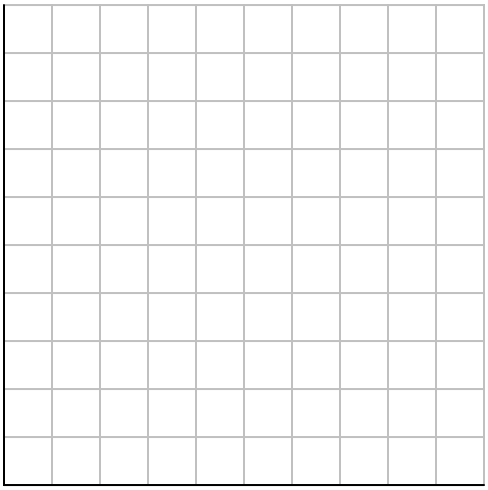
Summative Assessment

NAME_____DATE_____

1. ***Bacteria Problem*** A population of bacteria has been growing over the past 5 hours as shown in the table below:

Hour	0	1	2	3	4	5	***n***
Total# of Bacteria	4	12	36	108	324	972	

Graph the data.



a. What type of function models the data?

b. Write a rule to determine the total number of bacteria after ***n*** hours?

c. How many bacteria will there be after 8 hours? Explain how you determined your answer.

2. ***Interest Problem*** In order to come up with money to use for a dream vacation, Maria invested her two months salary in a lending company. It has a total of $11, 345. She agreed to the interest rate of 8% compounded continuously. What is the total amount of her money after ¾ year?

3. ***True or False***

a. The graph of every exponential function $f(x) = a^x, a > 0, a \neq 1,$ will contain the points (0, 1) and (1, a).
b. The graph of an exponential growth function $f(x) = a^x$ has both x-intercept and a y-intercept.
c. The y-axis is an asymptote for the graph of an exponential growth function $f(x) = a^x$
d. The equation $y = (\frac{1}{10})^x$ is an example of an exponential growth function.
e. In exponential growth, the proportional amount of change in two time intervals of the same length is the same.

4. ***Short Answer*** Write a real-life experience where exponential growth can or has occurred.

_____

_____

_____

_____

_____

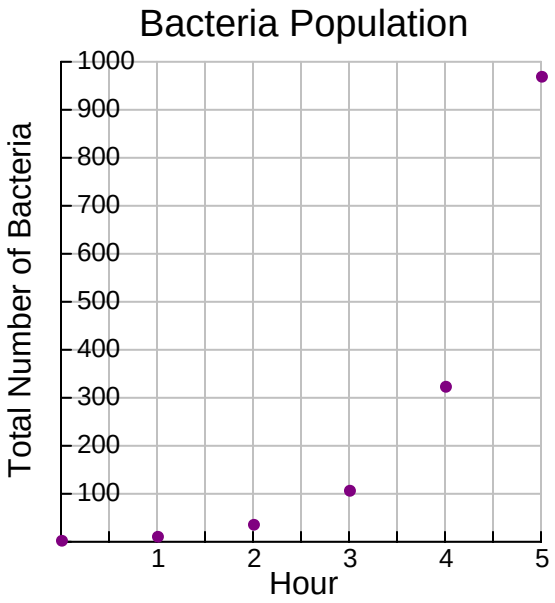
_____

SUMMATIVE ASSESSMENT ANSWER KEY

1. Bacteria Problem A population of bacteria has been growing over the past 5 hours as shown in the table below:

Hour	0	1	2	3	4	5	n
Total# of Bacteria	4	12	36	108	324	972	

Graph the data.



a. What type of function models the data?

Exponential function

b. Write a rule to determine the total number of bacteria after n hours?

As the time increases on integer values increasing by 1, the values of the number of bacteria form a geometric sequence with constant ratio 3. Using the formula $y = 4 \bullet 3^x$***, the total number of bacteria after*** **n** ***hours will be*** $y = 4 \bullet 3^n$**.**

c. How many bacteria will there be after 8 hours? Explain how you determined your answer.

26, 244 bacteria, using the formula $y = 4 \bullet 3^x$ ***and substituting the value of x that is 8,*** $y = 4 \bullet 3^8$***. Evaluating the expression, we determine that the number of bacteria is 26, 244 after 8 hours.***

2. ***Interest Problem*** In order to come up with money to use for a dream vacation, Maria invested her two months salary in a lending company. It has a total of $11, 345. She agreed to the interest rate of 8% compounded continuously. What is the total amount of her money after ¾ year?

$$A = Pe^{rt} \quad where \quad P = 11345 \quad r = 0.08 \quad t = 0.75$$
$$A = 11345e^{.08(.75)}$$
$$A = 11345e^{.06}$$
$$A = 12046.54$$

3. ***True or False***

a) The graph of every exponential function $f(x) = a^x, a > 0, a \neq 1,$ will contain the points (0, 1) and (1, a).

True

b) The graph of an exponential growth function $f(x) = a^x$ has both x-intercept and a y-intercept.

False. Exponential curves have no x-intercepts.

c) The y-axis is an asymptote for the graph of an exponential growth function $f(x) = a^x$.

False. The x-axis is an asymptote of exponential curve.

d) The equation $y = (\frac{1}{10})^x$ is an example of an exponential growth function.

True

e) In exponential growth, the proportional amount of change in two time intervals of the same length is the same.

True

4. ***Short Answer*** Write a real-life experience where exponential growth can or has occurred.

Answers may vary.

Authors:

Dileep S. Rajan
Theodore Roosevelt SHS
District of Columbia Public Schools

Marcelina Pascual
Theodore Roosevelt SHS
District of Columbia Public Schools

Jocylene S. Comia
Eastern SHS
District of Columbia Public Schools