MEMO ROUTING SLIP	*NEVER USE FOR APPROVALS, DISAPPROVALS, CONCURRENCES, OR SIMILAR ACTIONS.*		
1 NAME OR TITLE S/Asst	INITIALS		CIRCULATE
ORGANIZATION AND LOCATION Mr Friedman	DATE		COORDINATION
2			FILE
			INFORMATION
3			NECESSARY ACTION
			NOTE AND RETURN
4			SEE ME
			SIGNATURE

REMARKS

FROM NAME OR TITLE NSA-314	DATE 8 June
ORGANIZATION AND LOCATION 17-117	TELEPHONE 60391

DD FORM 1 FEB 50 94 REPLACES NME FORM 94, 1 FEB 49, WHICH MAY BE USED. 16—56368-2 GPO

A METHOD FOR GENERATING IRREDUCIBLE POLYNOMIALS

A. Gleason
R. W. Marsh
NSA-314
26 May 1955

It was observed that, if the polynomial $f(x) = \sum_{i=0}^{P} \alpha_i x^i$ (coefficients in GF(2)) is irreducible and its root has maximum period $2^P - 1$, then the polynomial $F(x) = \sum_{i=0}^{P} \alpha_i x^{2^{i-1}}$ is irreducible and its root has maximum period. This was verified in all cases up to $P = 5$. We shall give a proof that F is always irreducible but leave unsettled the question of whether its roots are primitive.

Let K be a finite field of cardinal q (which must be a prime power and, in the case of particular interest, is 2). Let K* be a minimal algebraically closed field containing K. For each positive integer n there is in K* a unique field K^n of degree n over K; $K^* = \bigcup K^n$. We may regard K* as an infinite dimensional vector space over K; then each of the fields K^n is a vector subspace.

Let α be the mapping $x \rightarrow x^q$ of K* into itself.

Lemma 1. $\theta \in K^n \leftrightarrow \alpha^n \theta = \theta$

Proof: The field K^n has q^n elements, and the $q^n - 1$ non-zero elements from a group under multiplication. By the theorem of Lagrange every element of this group satisfies the relation $\theta^{q^n-1} = 1$, whence every element of K^n satisfies $\theta^{q^n} = \theta$. This proves one half of the lemma.

The polynomial $x^{q^n} - x$ can have at most q^n roots in K*. Hence all of the roots are in K^n. This proves the second half of the lemma.

The mapping α is an automorphism of K^* since it evidently satisfies $\alpha(\theta\varphi) = \alpha(\theta)\alpha(\varphi)$ and $\alpha(\theta + \varphi) = \alpha(\theta) + \alpha(\varphi)$ because q is a power of the characteristic. We have seen that $\alpha\theta = \theta$ if $\theta \in K = K'$. Hence α is a linear transformation of K^* regarded as a vector space over K.

Lemma 2. If $\theta \in K^*$, the degree of θ is the least positive integer n for which $\alpha^n\theta = \theta$

Proof: Obvious from lemma 1.

Theorem: Let $f = \sum_{i=0}^{p} b_i x^i$ be an irreducible polynomial of degree p over K whose roots are primitive in K^p. Then $F = \sum_{i=0}^{p} b_i x^{q^i - 1}$ is an irreducible polynomial of degree $q^p - 1$.

Proof: Consider any root θ of F. Evidently $\theta \neq 0$. We have then $0 = \theta F(\theta) = \sum_{i=0}^{p} b_i \theta^{q^i} = \left(\sum_{i=0}^{p} b_i \alpha^i\right)\theta = f(\alpha)\theta$.

The set of all polynomials P such that $P(\alpha)\theta = 0$ is an ideal $\mathfrak{a}$ in the polynomial ring over K. Since this ring is a principal ideal ring and $\mathfrak{a}$ contains the irreducible polynomial f, $\mathfrak{a}$ is either the unit ideal or the principal ideal (f). The former possibility implies $\theta = 0$ which is false, so $\mathfrak{a} = (f)$. By lemma 2, the degree of θ is the least integer n for which $(\alpha^n - 1)\theta = 0$, that is, the least integer n for which $x^n - 1 \in (f)$. Since the roots of f are primitive this integer is $2^p - 1$.

The minimal polynomial for θ is therefore an irreducible polynomial of degree $2^p - 1$ which divides F. Comparing degrees we see that the quotient is in K, hence F is irreducible. q.e.d.

It may be remarked that in case f does not have primitive roots we can see that F splits into irreducible factors of degree equal to the order of the roots of f.

Concerning the second question as to whether F has primitive roots in the case K = GF(2), it may be remarked that if true we could then obtain an algebraic recursion giving only irreducible polynomials by iterating the procedure. Since this is closely related to a prime generating function, it is rather unlikely to be provable by elementary methods, if true. Starting with q = 3, K = GF(3), and the irreducible polynomial $x^2 - x - 1$ which has primitive roots we obtain the irreducible polynomial $x^8 - x^2 - 1$, whose roots have order 160, a far cry from being primitive. This also indicates that any proof would have to rely on number theoretic properties of the number 2.