Title: Staggered Starts: An Application of Arc Length

Brief Overview:

This is a series of three lessons. In the first lesson, students will be given track designs with varying dimensions and asked to determine starting positions for each lane using their knowledge of arc measure, circumference, arc length and the measurement of a central angle. They will be asked to write a brief report explaining their calculations. The second lesson builds on the concepts of lesson 1 and includes using both radian and degree measure. Using these ideas, students will design a track using their own dimensions, again determining the starting positions for each lane. The third lesson involves extending the concepts to elliptical shaped tracks and reviews the definition and properties of ellipses. Lesson 3 should only be taught after a unit on conic sessions including ellipses has been taught.

NCTM Content Standard/National Science Education Standard:

- Recognize and apply mathematics in contexts outside of mathematics
- Communicate their mathematical thinking coherently and clearly to peers, teachers, and others
- Build new mathematical knowledge through problem solving
- Understand patterns, relations, and functions
- Use mathematical models to represent and understand quantitative relationships
- Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships

Grade/Level:

Lesson 1 and 2: Geometry and above
Lesson 3: Algebra 2 and above

Duration/Length:

Three 90 minute lessons

Student Outcomes:

Students will:

- Given the dimensions and shape of a multi-lane running track, determine the starting positions for each lane.
- Write a brief report describing technique used to determine starting lines
- Design their own track of a given length, marking the starting positions for each lane on the track.
- Apply the properties of ellipses.
- Determine the approximate the lengths of major and minor axes given the circumference of an ellipse.

Materials and Resources:

- Graphing Calculator
- Handouts
- Protractors
- Straightedges
- Graph paper or Geometer Sketchpad

Development/Procedures:

Lesson 1

Pre-assessment – Worksheet includes finding circumference and arc length.

Launch – Present picture of running track that shows runners in a staggered start.
If possible, obtain a video of a race showing a staggered start.
Pose questions: "Why is the start staggered?"
"If a race is one or more laps, is it fair for all the runners to start at the same place?"
in order to generate discussion on staggered starts and distances. Relate discussion to pre-assessment worksheet.

Teacher Facilitation – Distribute "Staggered Starts-Circular Tracks". Discuss finding circumference and arc length as in worksheet in pre-assessment. Explain to students that runners will run as close to inside of the lane as allowed, therefore all measurements are done to the left side of any lane.

Student Application – Distribute"Staggered Starts-Standard Tracks – Day 1."
In pairs, students will determine the starting positions of runners in each lane on a standard shaped track design. Calculate radius of ends of track. Each additional lane will have a radius of $(100/\pi) + 2$. Discuss their results. The idea here is for students to apply the method for determining the starting positions on the circular track to that of the standard track. (Since it is on a semi-circle, it is the same). Determining the positions using central angles will be done in the next lesson. If time is short, they may complete this at home.

Embedded Assessment –Walk around and check responses and results of the calculations. Evaluate students description of process used to determine starting positions in student report.

Reteaching/Extension –

Reteaching - How did you determine the starting positions for runner? What were the calculations used? Would your description give the track operator enough information to place the starting lines precisely?

Extension - Is there a pattern in the different starting points? Think about the starting lines, would the starting positions be placed at differently if the race was 200 meters on a 400 meter track? What about a 300 meter track?

Lesson 2

Pre-assessment –Distribute "Warm-up Lesson 2" worksheet. It reviews central angles and arc lengths and changing between radian and degree measure.

Launch – Present picture of similar track used in Day 1. Specify the starting point and the distance of the race. How would placements of the starting lines be made?

Teacher Facilitation – Distribute "Staggered Starts – Lesson 2 Worksheet". Give students time to answer question one on worksheet. Lead students as to the method of measuring a distant on an arc, i.e. tape measure, ruler, or some other method. Lead discussion towards central angles and arc measure and the variation of angle θ for the different lanes. For item 5 on worksheet, teacher may choose to have all students design the same shape and size track or allow them to be creative and only choose the size you would like them to use.

Student Application – Realize relationship between central angles and arc measure and arc length. Find the central angle measurement for each of the lanes using both radians and degrees. Compute the arc length. Design track.

Embedded Assessment – Teacher will check the results of calculations of track design. Have students calculated the correct angles and arc length? Have the students placed their starting marks in the correct positions? Are their explanations feasible?

Reteaching/Extension –

Reteaching - For students with problems calculating the correct angles provide additional instructions and examples on finding angle measures given arc lengths or converting degrees to radians and vise versa.

Extension – Have students consider possibility of an elliptical shape track and determine the information needed to find starting position for 6 lane track.

Lesson 3

It is stressed that this lesson must be taught only after students have had a thorough grounding in the equation and properties of an ellipse.

Pre-assessment- Distribute worksheet on ellipses.

Launch – Discussion - What is an ellipse? What is the major axis? What is the minor axis? What are the foci? What information did the students decide would be needed to design an elliptical track?

Teacher Facilitation – Discussion on equations of ellipses. Distribute worksheet. Generate discussion about the use of calculus to arrive at the equation for the circumference of an ellipse.

Student Application – Using the equation for the circumference of an ellipse, and the fact that the circumference is going to be 400 meters, determine the shape of the ellipse. Students will determine some combination of a and b (major and minor axes). From this, they will find the focus points of the ellipse and then construct a picture of the graph.

Embedded Assessment – Exploration of major and minor axes, understanding of the reason behind the focus points.

Reteaching/Extension

Reteaching – Review properties of ellipses. Review equation of an ellipse and equation for foci.

Extension – Have students investigate how to get more accurate approximations of the angle measures required to determine the starting positions.

Summative Assessment:

Have students determine the starting positions and the finish line for a 200 meter race on a 400 meter standard shaped track.

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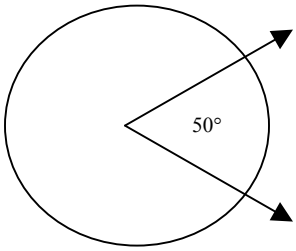
Staggered Starts Worksheet – Pre-activity Worksheet Day 1

What do you remember about circles and arcs?

1. Find the circumference of a circle with a radius of 3.5 cm.

2. Find the radius of a circle with circumference 35 m.

3. If a central angle of a circle measures 50°, what would be the measure of the intercepted arc?

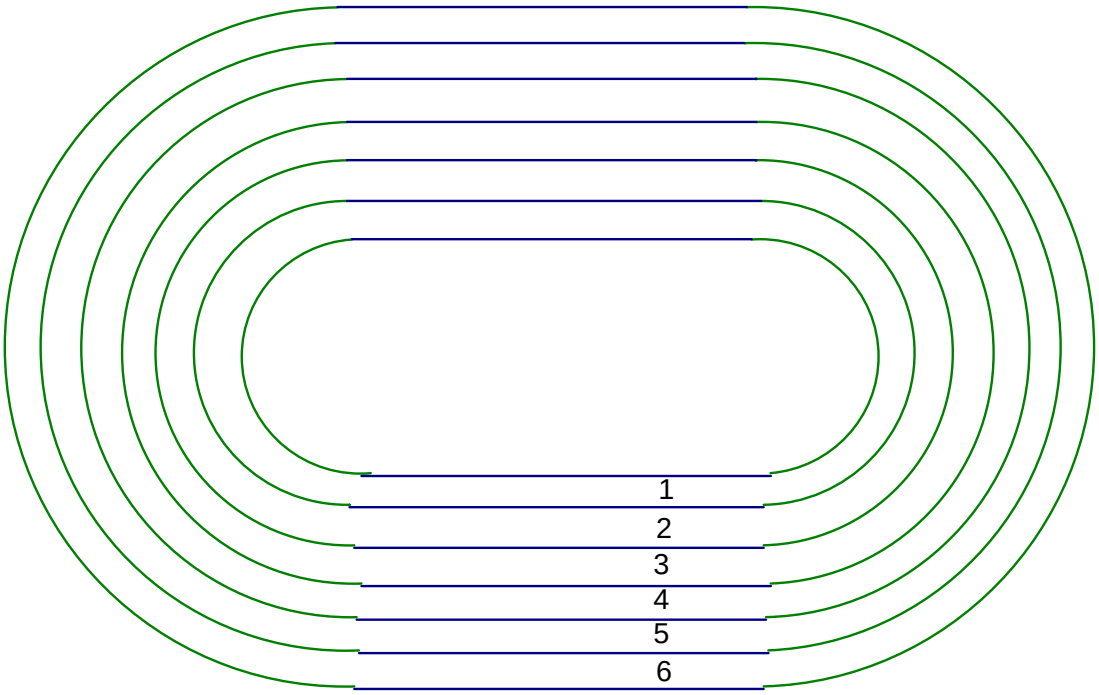


4. For a circle with a radius of 10 cm, what would be the length of a 30° arc?

Staggered Starts Worksheet – Standard Tracks - Day 1

Your next job is to determine the approximate starting positions for a standard shaped 400 meter track which is composed of two congruent straight-aways and two congruent semi-circles as shown below. One possible configuration is to have each straight-away and each inner-most semi-circle be 100 meters. The width of a lane continues to be 2 meters.

100 meters



100 meters

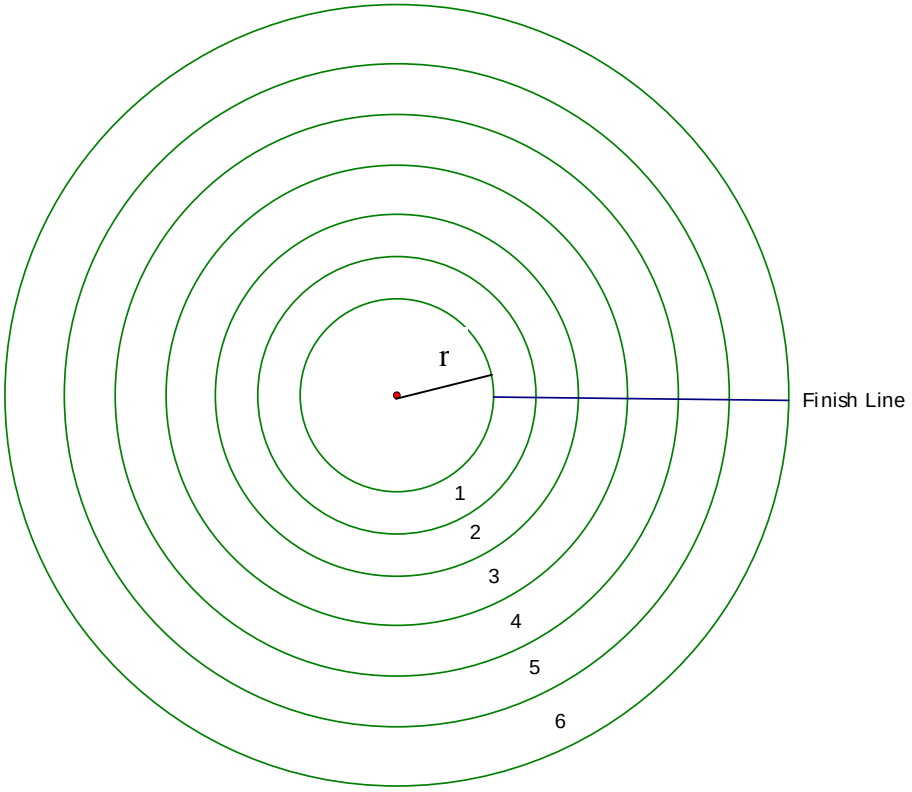
Determine where you want the finish line to be for a 400 meter race. Then determine the approximate starting positions for the runners in each lane. Mark the approximate positions on the diagram. Write a brief report that clearly explains how you arrived at your answer.

(The assumptions that the runners run counter-clockwise and run as close to the left of their lane as possible still hold.)

Staggered Starts Worksheet – Circular Tracks - Lesson 1

Suppose you work for a company that owns running tracks and your job is to determine the starting positions for the different track events. All of the tracks in this lesson are 400 meters which is standard in many competitions. The 400 meters is measured from the inside of lane 1. The width of a lane is 2 meters. Assume runners run counter-clockwise and as close to the left of their lane as they can.

The first track you are to deal with is circular and has six lanes as shown below.



1. This pattern is an example of _____ circles.
2. We know that if the runner in lane 1 completes one lap he will travel _____meters.
3. Given your answer to item 2, find the distance from the center of the track to the inside of lane 1 (the radius r).

4. Find the distance from the center of the track to inside of lane 2.

5. Calculate how far the runner in lane 2 would go if she completes one full lap. Show your calculation in the space below.

6. How much farther does the lane 2 runner travel than the lane 1 runner?

7. Calculate how far the runner in lane 3 would go if he completes one full lap. How much farther does the lane 3 runner travel than the lane 1 runner? Show your calculations.

8. Complete the table.

Lane	Radius	Distance for a full lap (m)	Distance over 400 meters
1			
2			
3			
4			
5			
6			

9. Should all the runners start along side each other? Why or why not?

Mark on your diagram the spot from where you think each runner should start.

10. What method might you use to place the starting lines for each lane precisely?

11. Do you see a pattern in the third column of the table?

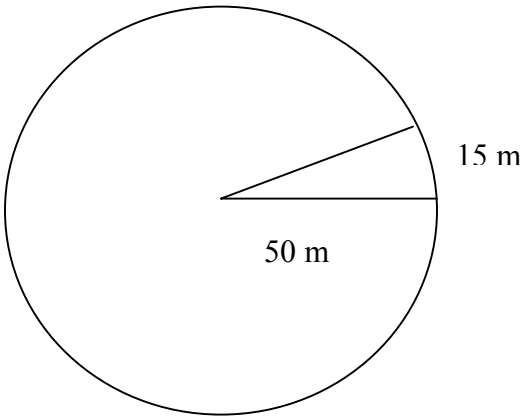
If the track had 12 lanes, use the pattern to predict by how much the runner in lane 12 would exceed 400 meters.

Staggered Starts – Warm-up – Lesson 2

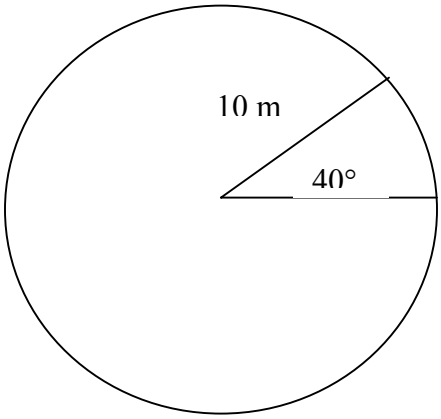
1. 2.4 radians is equivalent to how many degrees?

2. 80° is equivalent to how many radians?

3. Find the measure of a central angle that intercepts an arc of length 15 meters on a circle with a radius of 50 meters.



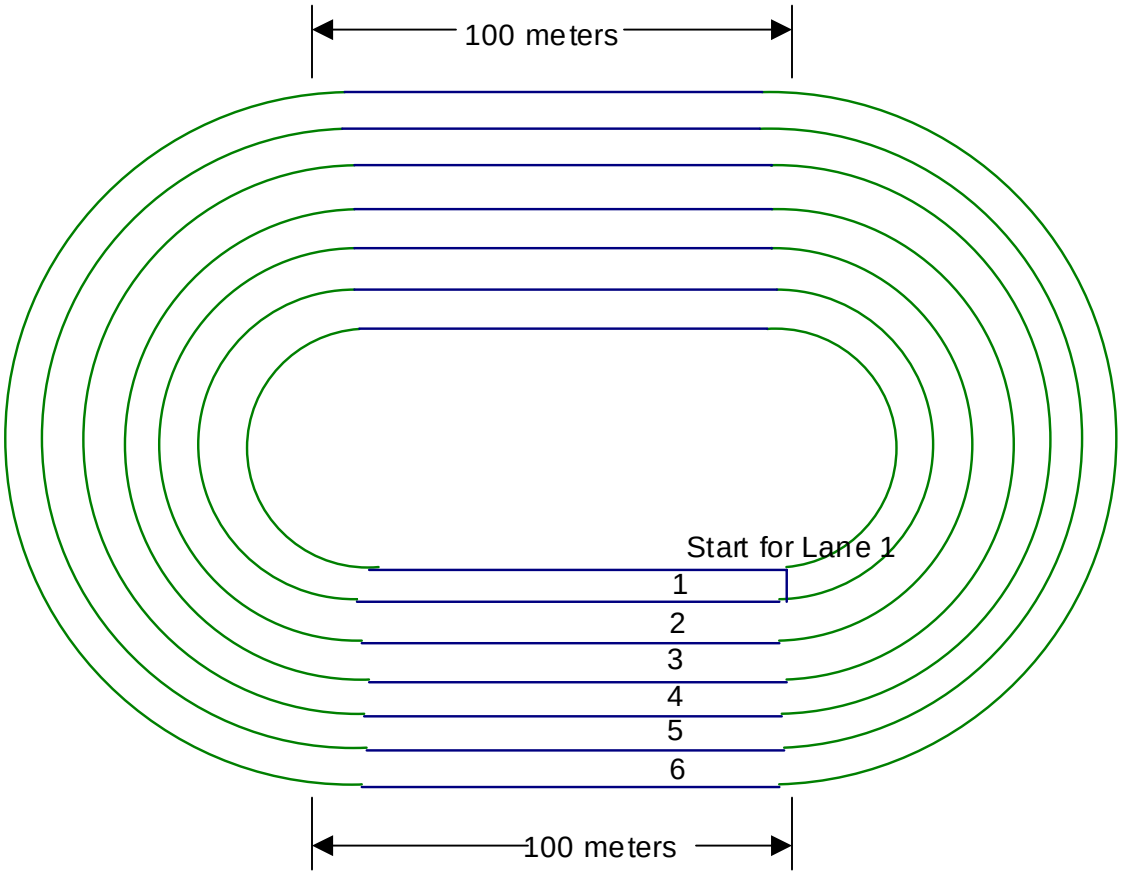
4. How long is an arc of a circle with a radius of 10 meters formed by a central angle of 40°?



5. On a 400 meter circular track how far does the runner in lane 2 go if he runs one full lap? (See Circular Tracks Worksheet if you need to do so).

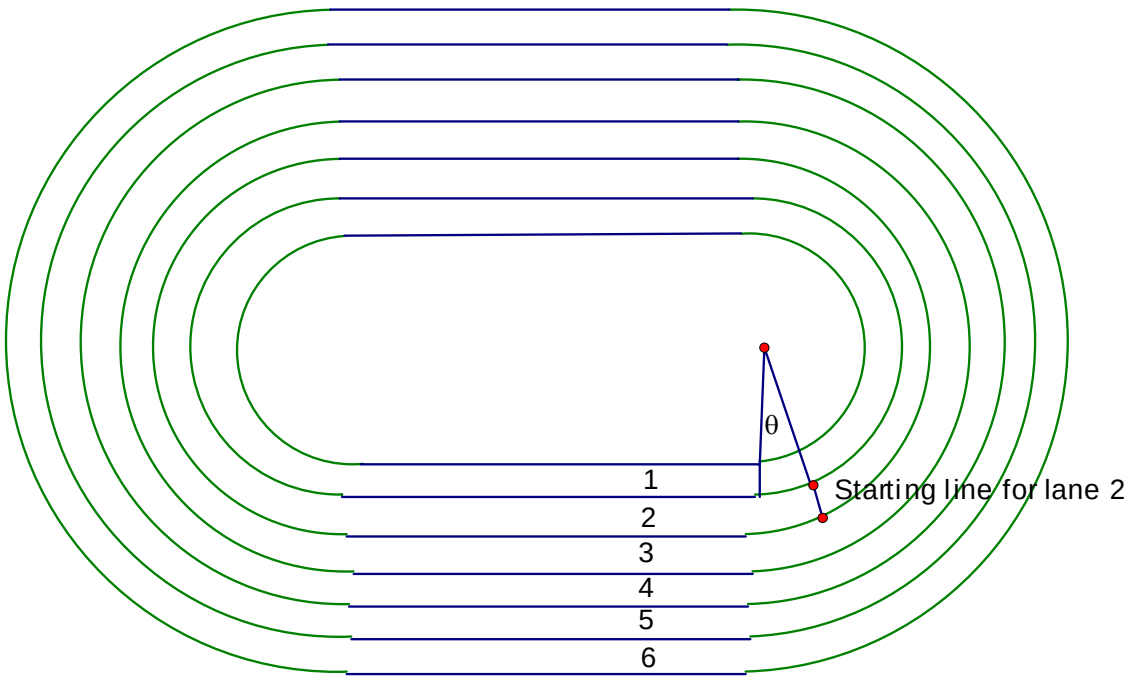
Staggered Starts- Lesson 2 Worksheet

In the standard-shaped 400 meter track that we dealt with in the last lesson, the starting position is specified as shown below. The runners run counterclockwise. The question for today is, "how can we place the starting lines precisely in each lane given that they are on an arc?"



1. We now know that the runner in lane 2 needs to start 4π or approximately12.57 meters ahead of the runner in lane 1. (See items 6 and 7 from Circular Tracks Worksheet). How should we mark off that distance?

2. Approximate where the center of the semi-circle would be and create a central angle θ as shown.



3. Find an expression for the measure of θ for lane 2. Give your answer in both radians and degrees.

4. Complete the table for all the lanes:

Lane	Arc length	θ (radians)	θ (degrees)
1	0	0	0
2	12.57		
3			
4			
5			

6			

5. Design either your own 6-lane 400 – meter standard shaped track in which the straight-aways are not 100 meters or design a standard-shaped track of a length other than 400 meters. Show your finish line and starting lines for each lane. Give directions so that a track official would know exactly where to place the starting lines.

Staggered Starts- Lesson 3 Warm-up

1. What is an ellipse?

2. Using graph paper, draw an ellipse with center at (0,0) and with a vertical minor axis of length 6 units and a horizontal major axis of length 8 units.

3. Write the equation of your ellipse in item 2.

4. Plot the foci of your ellipse.

5. Solve $800 = 10\sqrt{\frac{a^2 + 100}{2}}$

Staggered Starts- Lesson 3 – Elliptical Tracks

Running tracks are sometimes elliptical. In this lesson, we will consider the starting positions for a 400 meter elliptical track with 2 meter wide lanes.

Although calculus is required to find the circumference and arc length of an ellipse, the circumference can be approximated by the equation

$$C = 2\pi\sqrt{\frac{a^2 + b^2}{2}}$$

where $2a$ is the length of the major axis and $2b$ is the length of the minor axis.

1. Determine a possible pair of values for your a and b such that the circumference would be 400 meters using the above equation.

2. Draw an approximate scale model of this ellipse on your graph paper to represent lane 1 of your track. Choose a location for the finish line.

3. Determine approximately where the foci would be.

4. Will the arc lengths that determine the distances between the runners be the same as they were for the circular track and the standard-shaped track?

5. What point or points do you think the track official will use to determine the exact starting positions for each lane?

6. Look at the angle measures you used to determine the starting positions on the standard track. Write a brief description of how the angles to determine the starting positions on an elliptical track might differ from those for the standard-shaped track.

7. Would your answers to item 6 be the same for any other locations of the finish line? Why or why not?

Answer Key – Pre-activity Worksheet Day 1

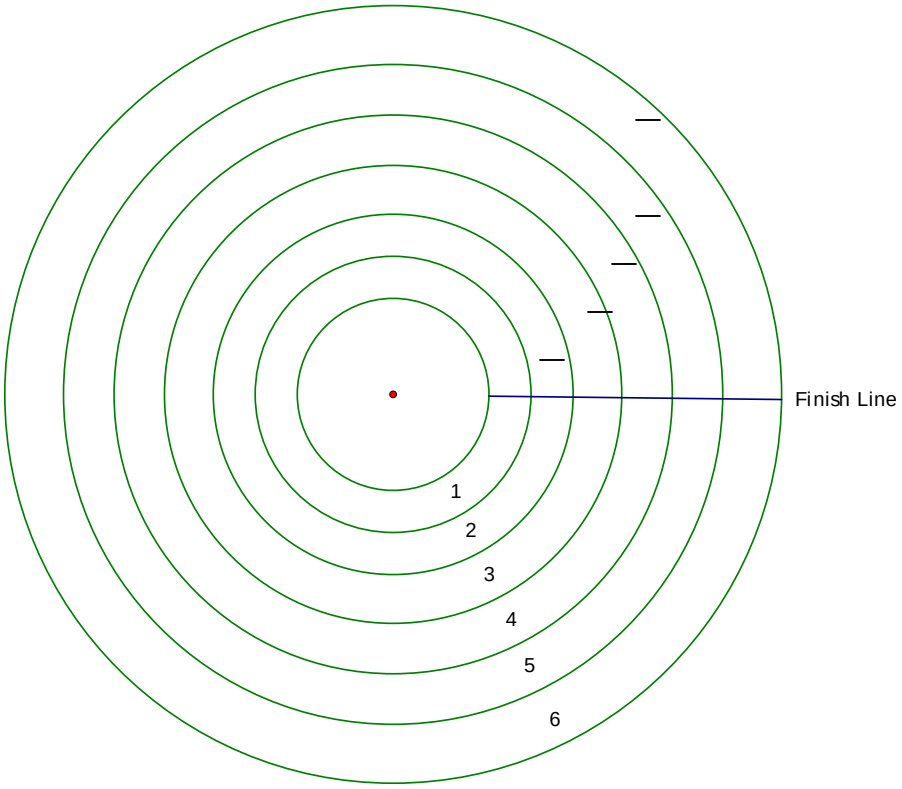
1. ~21.99 cm
2. ~5.57 m
3. 50°
4. ~5.24 cm

Answer Key –Worksheet - Circular Tracks Day 1

1. concentric
2. 400 meters
3. $\frac{200}{\pi} \approx 63.66$ m
4. ~65.66 m
5. ~412.57 meters
6. ~12.57
7. ~425.13
8.

Lane	Radius (m)	Distance for a full lap (m)	Distance over 400 meters
1	~63.66	400	0
2	~65.66	412.57	12.57
3	~67.66	425.13	25.13
4	~69.66	437.70	37.70
5	~71.66	450.27.	50.27
6	~73.66	462.83	62.83

9. No. Runners in outside lanes have farther to go to complete a lap.



One possible set of starting lines is shown.

10. Two possible responses:
    - Use a ruler or tape measure and bend around the curve.
    - Use central angles to achieve the desired arc length.

11. Pattern is 4π (n-1) where n is the lane number.
    Runner in lane 12 would exceed 400 meters by 138.2 meters.

Answer Key – Standard Tracks Day 1

Report should include:

- Diagram with finish line and reasonable starting positions shown.
- The distances between the starting line for lane 1 should be calculated correctly and the calculations shown. For lane 1, the radius is still $100/\pi$ because it is a semicircle with a circumference off 100 meters. The radius for lane 2 will be $(100/\pi) + 2$ since the length of the two straightaways is 200m, the circumference of lane 2 is $200 + 2\pi((100/\pi) + 2) \approx 412.57$ and thus the arc length is still 12.57 (or 4π) meters. The distances for the other lanes are calculated the same way and the result is the same as those for the circular tracks. Lane 3: ~25.13 meters; Lane 4: ~37.70 m; Lane 5: ~50.27 m; Lane 6: ~62.83 m.

- Some explanation for their placements: Sample. I put the starting line at the beginning of a straight-away because it is easier to mark the distances with a straight line instead of around a curve.

Answer Key – Warm-up Lesson 2

1. ~137.5°
2. ~1.39 radians
3. ~0.3 radians or ~17.2°
4. ~ 7.0 meters
5. ~412.57 meters

Answer Key – Lesson 2 Worksheet

1. Answers will vary. Students should be led to the idea that central angles of the semi-circles need to be calculated from the arc lengths.

3. $\theta \approx .37$ radians $\approx 21.3°$

4.

Lane	Arc length (meters)	θ (radians)	θ (degrees)
1	0	0	0
2	12.57	.37	21.3
3	25.13	.70	40.2
4	37.70	1.00	57.1
5	50.27	1.26	72.3
6	62.83	1.50	86.1

5. Answers will vary.
   Sample: If the two straight-aways are 150 meters and each semi-circle is 50 meters, the arc lengths from the starting line in lane 1 and the central angles would be as shown below.

Lane	Arc Length	θ	
1	0		
2	12.56	.70	40.2
3	19.91	1.26	72.3
4	21.91	1.72	98.6
5	23.91	2.10	120.5
6	25.91	2.43	138.9

Answer Key – Warm-up Day 3

1. The set of points such that the sum of the distances from each point to two points called the foci is constant. Student answers will probably include an "oval" shape.

2. Graph should pass through (4,0); (-4, 0); (0,3); and (0,-3) and should be reasonably elliptical.

3. $\frac{x^2}{16}+\frac{y^2}{9}=1$

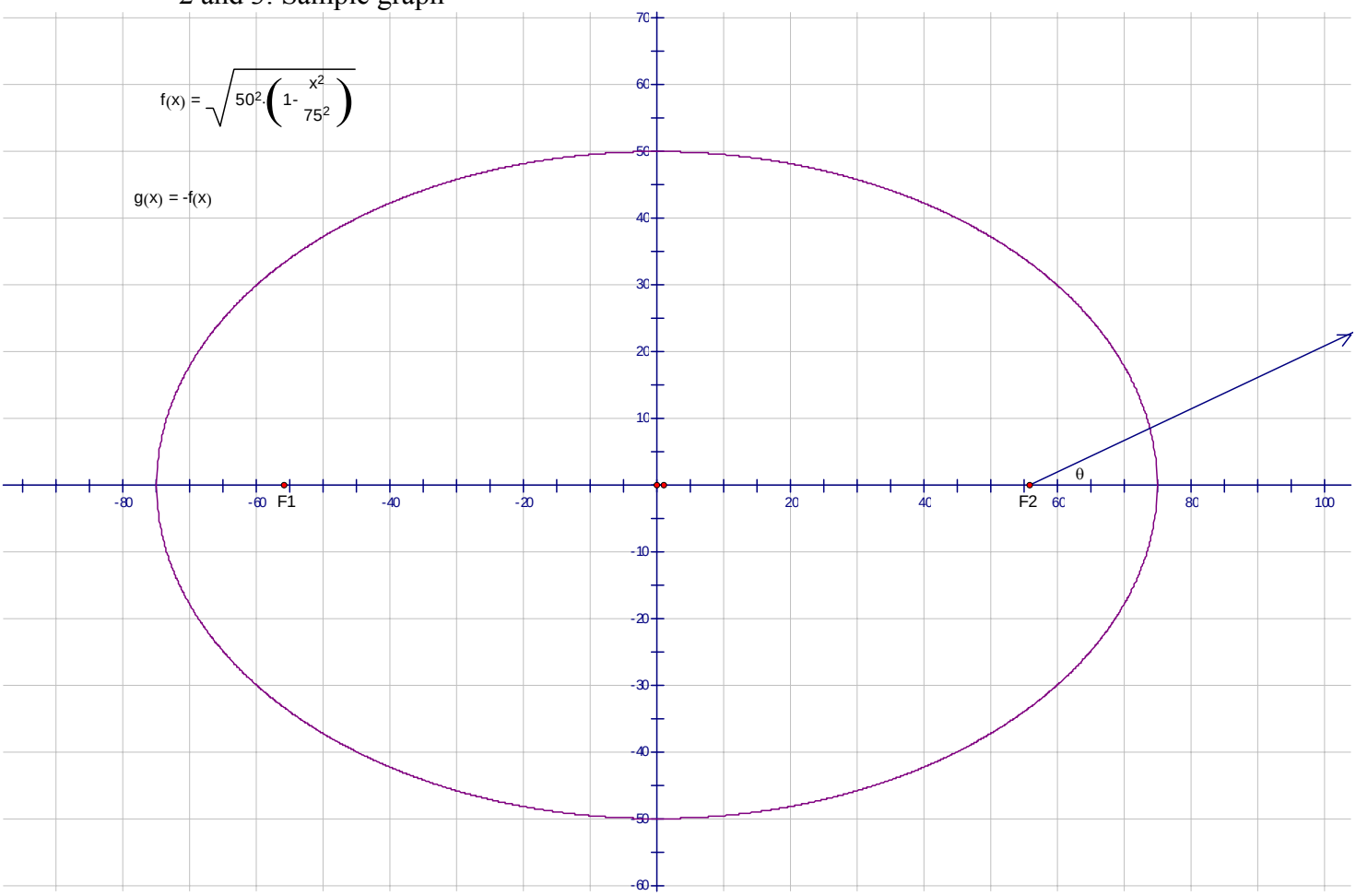
4. $(\sqrt{7},0);(-\sqrt{7},0)$

5. 112.69

Answer Key – Day 3 Elliptical Tracks

1. Sample: $a = 75$, $b = 50$. This yields an approximate circumference of 400.48 meters.

2 and 3: Sample graph



Finish line could be anywhere the student chooses.

4.Yes

5. The foci.

6 and 7. Will depend on where the finish line will be.

If the finish line and the start line for lane 1 is placed at (75,0), and the vertex of θ is at F2 as shown , then the angle for lane 2 will be smaller than it was for the circular or standard –shaped track. The angle measures will increase at a greater rate from the innermost lane to each succeeding outer lane than they do for the

circular or standard-shaped track. If the finish line were placed at (0,50), then the angles for lane 2, lane 3, etc. would start out greater than they do for the circular or standard shaped track and then increase at a smaller rate.