Title: Pythagoras Wasn't Just a Square!

Brief Overview:

While most students are familiar with the Pythagorean Theorem ($a^2 + b^2 = c^2$), many don't realize why or how it works. With knowledge of similar figures, students can explore how the areas of any similar figures drawn on right triangles can be used to find a missing side. Using Geometer's Sketchpad software and the resource materials contained in this unit, the teacher can take Geometry students through some interesting applications of the Pythagorean Theorem.

NCTM Content Standard/National Science Education Standard:

- Represent problem situations with geometric models and apply properties of figures.
- Classify figures in terms of congruence and similarity and apply these relationships.
- Deduce properties of, and relationships between, figures from given assumptions.
- Translate between synthetic and coordinate representations.
- Deduce properties of figures using transformations and using coordinates.

Grade/Level:

- Grade 9/10/11 Geometry

Duration/Length:

Three 50-minute lessons with Extension/Homework

Student Outcomes:

Students will:
- Represent geometric figures using Geometer's Sketchpad.
- Use a variety of shapes (other than squares) to find the missing sides of right triangles.
- Construct a table of values to investigate mathematical patterns evident in the Pythagorean relationship.
- Determine the maximum volume of a cone made from a circle with a fixed radius and investigate its relationship to the Pythagorean Theorem.

Materials and Resources:

- Geometer's Sketchpad (2001 KCP Technologies Inc., http//:keypress.com/sketchpad) **OR** Geoboards OR dot paper.
- Overhead Transparencies

- Copies of Resources 2, 5, 6,
- Markers
- Construction paper
- Scissors
- Resource Materials

Development/Procedures:

Lesson 1

Preassessment – Warm Up (Resource 1)

Launch – The discussion in the warm-up will assure the students know the definitions of the vocabulary that will be used in Geometer's Sketchpad. Students will create these figures in Sketchpad while they learn to use the interface.

Teacher Facilitation – The instructions for this activity (Resource 2) can be copied for students or explained and demonstrated by the teacher.

Student Application – Students will follow directions to become familiar with the program. This activity is necessary if students have never used Geometer's Sketchpad.

Embedded Assessment – As students are following instructions the teacher can observe their computer screens. Since this is an introductory activity, it is important all students can use the software before moving to lesson 2.

Reteaching/Extension – "On your own" activities at the end of the instructions in Resource 2 can challenge students to discover more about the software or reinforce their skills.

Lesson 2

Preassessment – Warm-up activity to activate prior knowledge of similarity and ratio. (Resource 3).

Launch – Remembering the Pythagorean Theorem as ($a^2 + b^2 = c^2$), students have probably used the formula, but may not have seen the areas proof. Follow the instructions on the resource to make a puzzle that shows how the areas off of the legs add together to make the area off of the hypotenuse. (Resource 3).

Teacher Facilitation – Following the instructions in Resource 4, use Geometer's Sketchpad to demonstrate alternative shapes that informally prove the Pythagorean Theorem.

Student Application – Students will follow the teacher's directions to create the diagrams on Sketchpad. If the program is not available, students using Geoboards or dot paper can create similar diagrams. The students should answer questions on their worksheet as they complete the activity. (Resource 5)

Embedded Assessment – As students to try and create their own shape from each side of the triangle. If they understand the concept of the two smaller areas being equal to the larger area, then they will be able to come up with similar figures. They can accomplish this through trial and error or by understanding the proportionality of the sides.

Reteaching/Extension –
Worksheet (Resource 6)

Lesson 3

Preassessment – Warm-up activity where students calculate the areas of shapes drawn on the legs of a right triangle. (Resource 7).

Launch - The teacher should explain to the students that today's investigation is a continuation of yesterday's. Students will compare area models that have been dilated to show that as long as the three shapes off of a right triangle are similar to each other, the area model works.

Teacher Facilitation – (Resource 8) If time doesn't permit the students to create the investigation on Sketchpad, this can be set up ahead for demonstration purposes. Follow the directions on Resource 8 to create a figure that can be dilated and record data in a table.

Student Application – Students complete the table and answer questions. (Resource 9)

Embedded Assessment – The students' ability to answer the questions based on the investigation shows that they understand how the area model works.

Reteaching/Extension

Summative Assessment:

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(Resource 1)

Warm Up – Day 1

1. Define the following geometric figures:

Line_____________________
Point_______________________
Segment______________________
Vertex_______________________

2. Describe the difference between *drawing* and *construction*:

3. Find the area of a rectangle with a length of 20 and a width of 12:

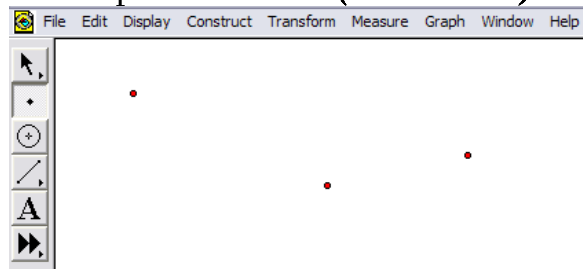
Using Geometer's Sketchpad ***(Resource 2)***

Create a point

Open Sketchpad and choose the point tool (under the arrow).

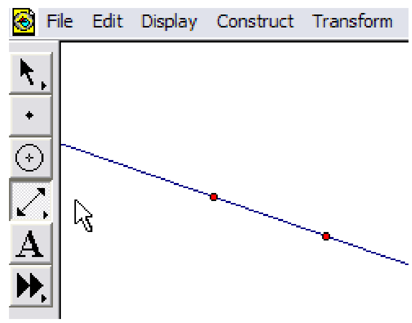
When you click on the blank screen, a point will appear.

The point will be highlighted in red, and you can deselect using the pointer tool (top of toolbar) and clicking anywhere on the screen.



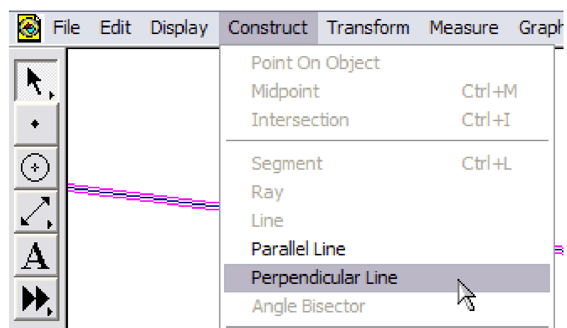
Construct a line through a point

1. Create a point.
2. Go to the Line/Segment tool (see right) and click and hold until the line symbol appears.
3. Click the point and it will highlight in blue. As you move the mouse the line will appear. When you click again the line will freeze.



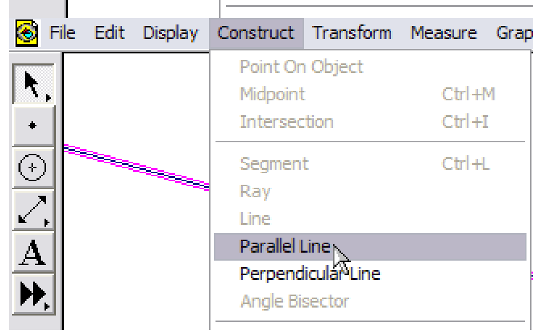
Construct a perpendicular line

1. Create a line.
2. Select the line and one point with the pointer tool.
3. Use the pointer to select **Construct**→Perpendicular Line.

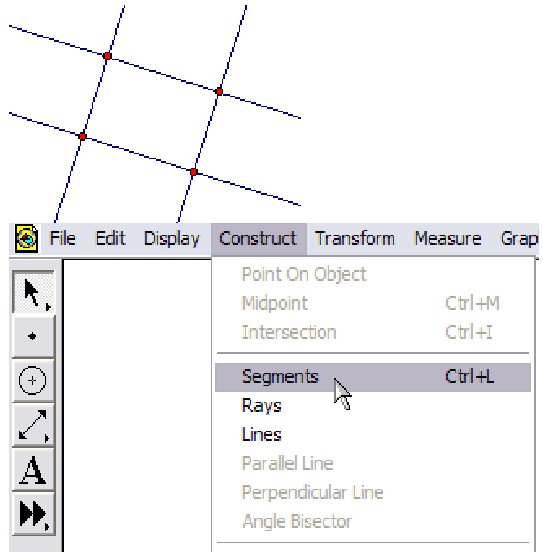


Construct a parallel line

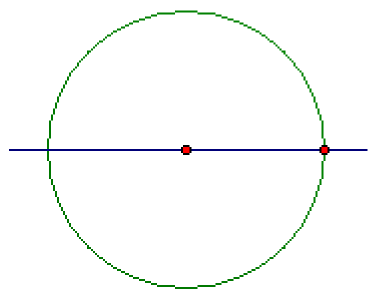
1. Create a line.
2. Create a point not on the line.
3. Select the line and the point.
4. Use the pointer to select **Construct** →Parallel Line.



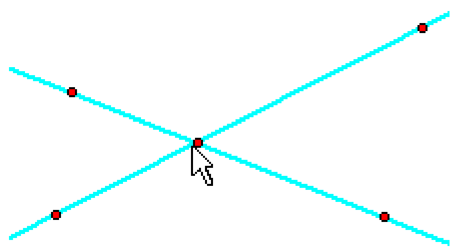
Construct a rectangle
1. Use parallel and perpendicular lines to construct a rectangle.
2. Select only the lines (not the points) and select **Display**→Hide Lines.
Construct a rectangle continued:
Note: When you use the point tool to create points on a line, roll over the line until it turns blue, then create the point.
3.Select the four points in either a clockwise or counterclockwise direction.
4. Select **Construct**→Segments



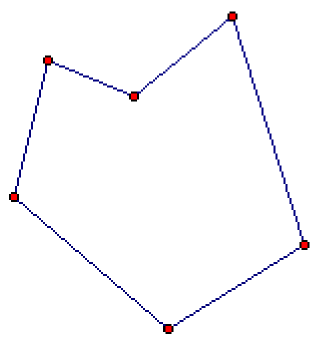
Construct a circle on a line:
1. Construct a line.
2. Use the circle tool and roll the pointer over one point in the line until it highlights. Click.
3. As you move the mouse a circle will appear. When you click again the circle will be constructed.



Now you try: Construct a point on an intersection:

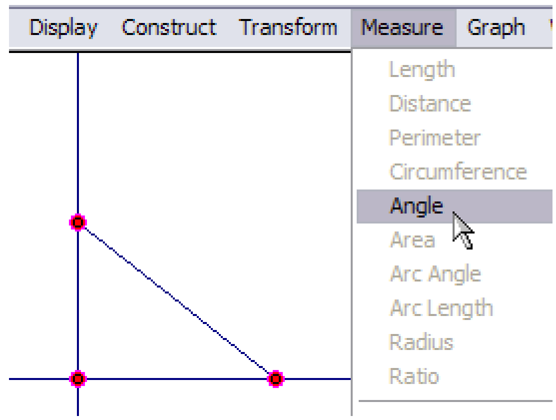


Construct the shape at the right using points and segments:



Construct a right triangle. Verify that your triangle is right by measuring the angle:

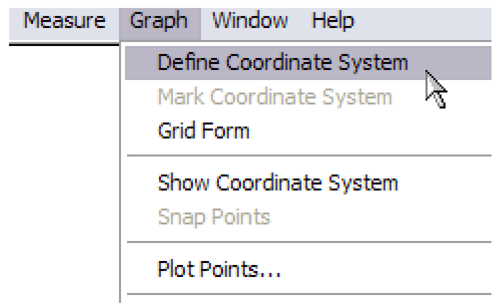
1. Select the points in the order that you would name the right triangle.
2. **Select Measure**→Angle.
3. The measure should read 90 degrees. (Sometimes the measures are off by a fraction of a degree because of the limitations of the display).



Use a coordinate grid:

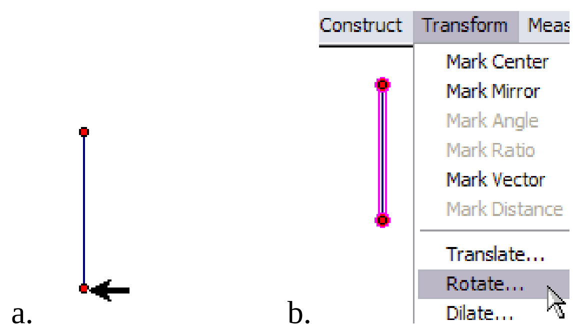
Select **Graph**→Define Coordinate System.

To enable the pointer to "snap" to the grid, select **Graph**→Snap Points.



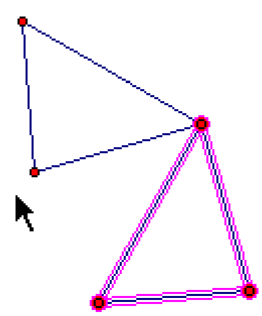
Transformations:

1. Construct a segment.
2. Double click the center of rotation. You should see a "bull's eye" when you do this. (a)
3. Select the entire figure and click **Transform**→Rotate.
4. Enter a degree measure for the rotation.



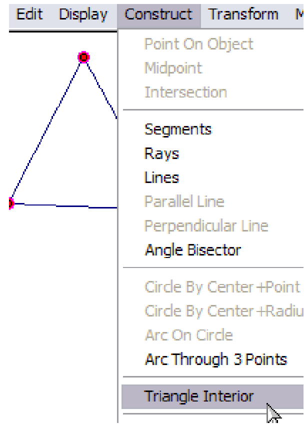
You try: Construct a triangle and rotate it 90 degrees counterclockwise around the right most point.

Note: Entering 90 degrees will rotate the object counterclockwise. To rotate clockwise you must enter 270 degrees.



Construct an interior:

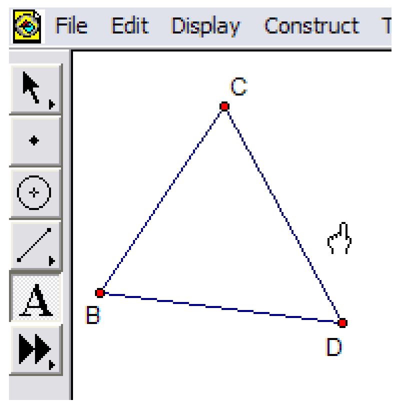
1. Construct a polygon.
2. Select each vertex in the order you would name them.
3. Select **Construct**→Triangle Interior.



Label your diagrams:

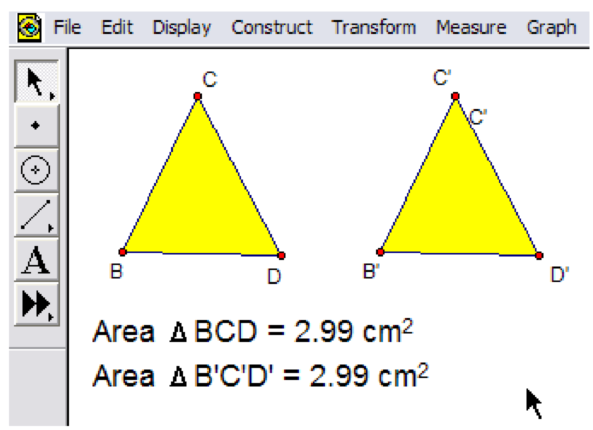
Using the text tool, click on a point and Sketchpad will label it.

Double click if you would like to change the label.



Put it together:

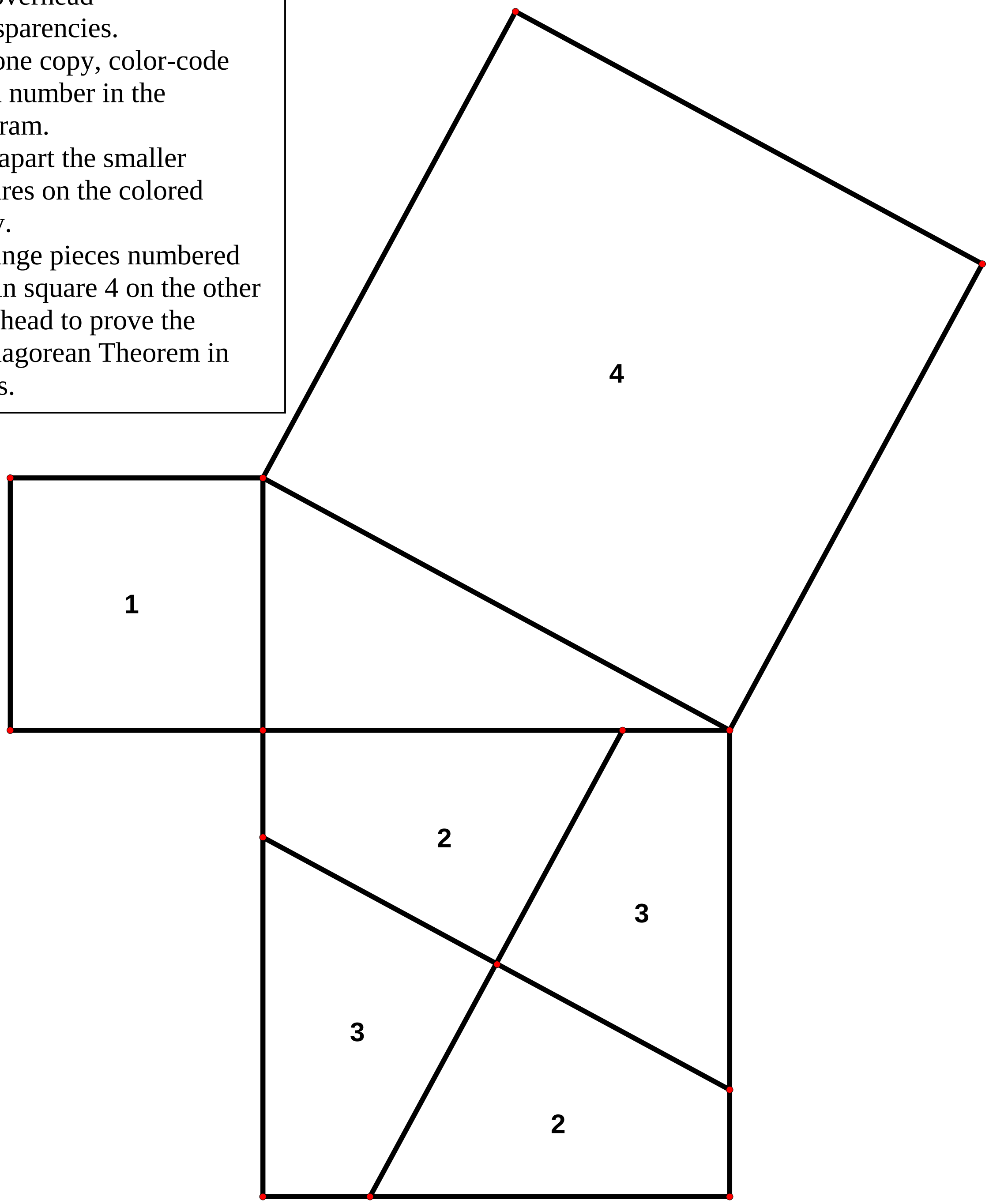
1. Create a polygon and label it.
2. Translate the polygon.
3. Construct interiors.
4. Select one interior and then select **Measure**→Area. Repeat for the other polygon.
5. Verify that the measures are the same.



Warm Up 1 *(Resource 3)*

Instructions:

1. Copy this graphic to two (2) overhead transparencies.
2. On one copy, color-code each number in the diagram.
3. Cut apart the smaller squares on the colored copy.
4. Arrange pieces numbered 1-3 in square 4 on the other overhead to prove the Pythagorean Theorem in areas.



Introductory Sketchpad Activity *(Resource 4)*

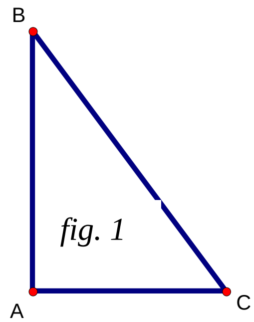


fig. 1

1. Open Geometer's Sketchpad. Select **Edit** → Preferences. Set the measures to **units**.
2. Select **Graph** → Show Grid.
3. Select **Graph** → Snap Points.
4. Use the point tool to create vertices A, B, and C, so that segment AB is vertical and AC is horizontal.
5. Select the three points and **Construct** → Segments *(fig. 1)*.

6. Double click A to mark it as a center of rotation.
7. Select segment AB and its endpoints.
8. Select **Transform** → Rotate, set the angle of rotation to 90 degrees *(fig. 2)*.

fig. 2

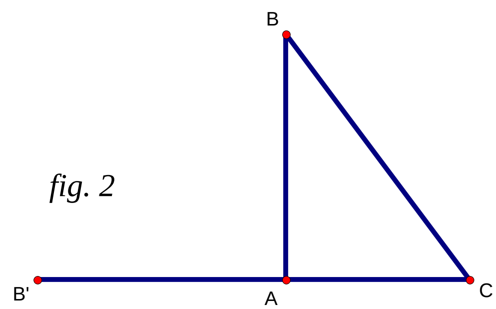
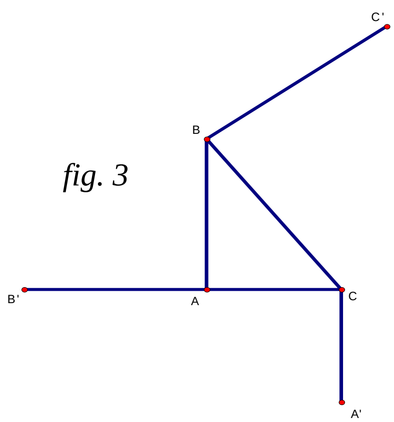


fig. 3



9. Repeat steps 6-8 on the other leg and the hypotenuse of the triangle *(fig. 3)*.

**Note: This figure will be used again in steps 16 and 17.*

10. Continue to rotate segments until you have formed squares on all sides of the triangle *(fig. 4)*.

11. Select all four vertices of one of the squares.
12. Select **Construct** →Construct Quadrilateral Interior.
13. Repeat for the other squares, changing colors in the **Display** menu if you prefer.
14. Select the interior of a square and select **Measure** → Area *(fig. 5)*

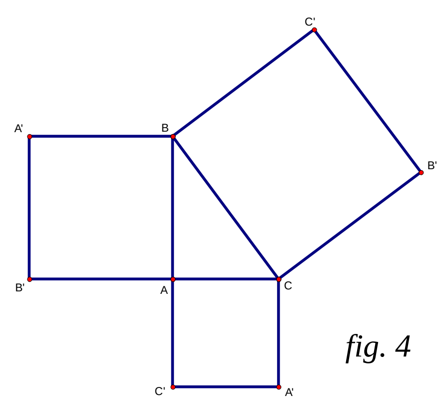
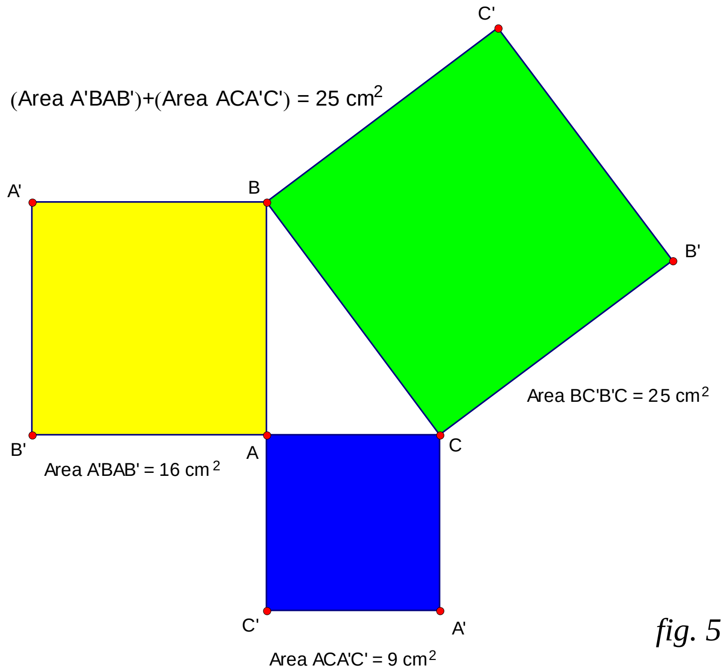


fig. 4

(Resource 4 continued)



15. Choose **Measure** → Calculate, add the measurements of the two smaller squares to show that their sums are equal to the area of the largest square *(fig. 5).*

fig. 5

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16. Begin with fig. 3 and create similar triangles by constructing segments B'B, A'A and C'C. Create the triangle interiors and measure and compare the areas (as in step 15).

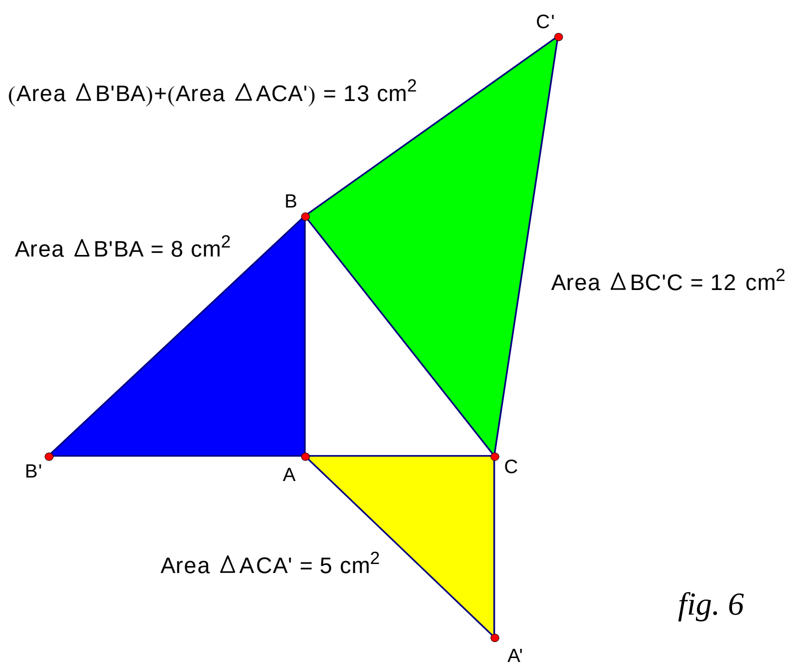

fig. 6

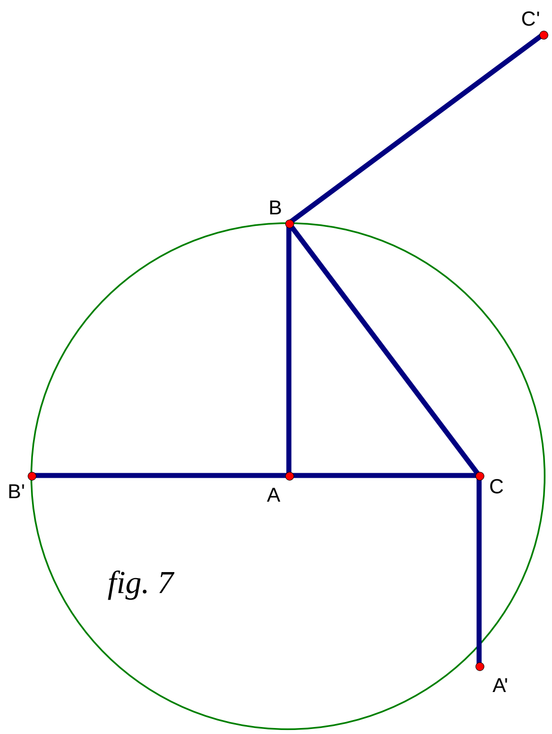

fig. 7

You can even try this with sectors of a circle. Begin with *fig. 3*:

17. In order, select point A and then point B.
18. Select **Construct** → Circle By Center + Point *(fig 7).*
19. In a **counterclockwise** direction, choose in order, points A, then B, then B'.
20. Select **Construct** → Arc On Circle. The arc should appear highlighted.
21. Select the circle somewhere that is not on the arc. Select **Display** → Hide Circle. Just the arc should remain.
22. Repeat for the other two sides, remembering to select points in a **counterclockwise** direction to form the arc.

23. Select one of the arcs (no points) and **Construct** → Arc Interior → Arc Sector.
24. Repeat for the other arcs *(fig 8).*
25. Use **Calculate** to find the sum of the areas of the smaller sectors and show that the sum equals the largest sector.
26. Have students create their own similar shapes on the legs and hypotenuse of a right triangle.

Note: This could be prepared ahead of time by the instructor as a demonstration, or used as a classroom lesson to teach students how to use Sketchpad.

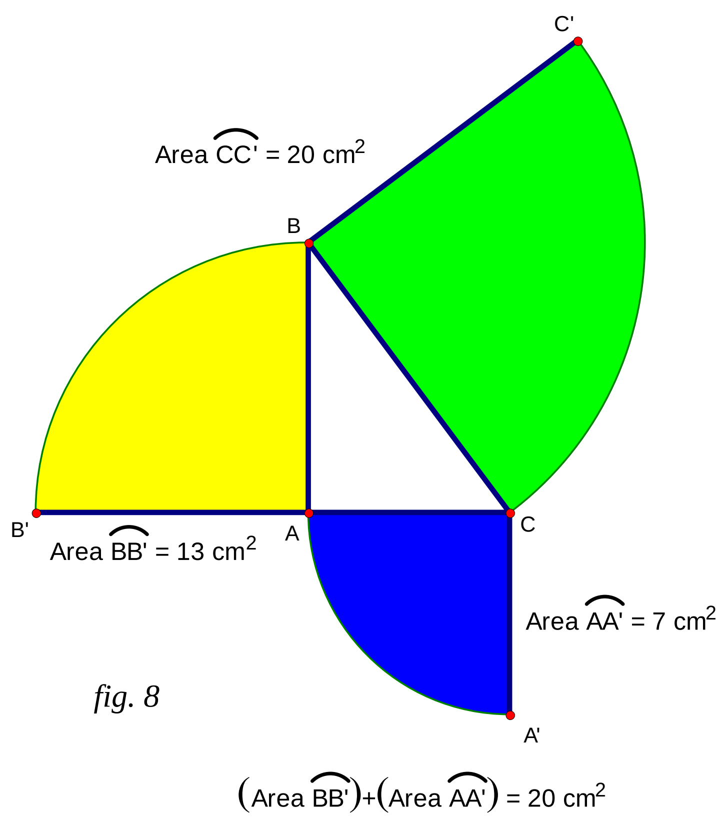

fig. 8

(Resource 5)

Pythagoras Wasn't Just a Square

1

Follow your teacher's instructions to form a right triangle with squares on the sides. Sketch your drawing to the left including measurements. How does this relate to the formula for the Pythagorean Theorem?

____

____

____

____

2

Sketch the diagram with the three triangles drawn on each side of the right triangle. Do the smaller triangles' areas add up to the same number as the triangle drawn on the hypotenuse?

____

____

Write an equation to show the addition of the areas based on the formula for the area of a triangle:______________________________

3

Draw a quarter-circle or another shape on the sides of a right triangle and sketch your diagram to the left. Calculate the areas of each shape and write your equation:

____

____

Why do you think the Theorem uses squares and the equation $a^2 + b^2 = c^2$ to calculate the missing side of a right triangle?

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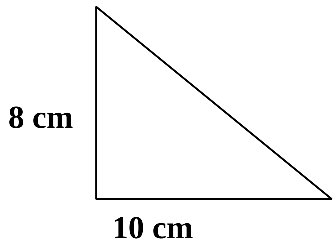
(Resource 6)

Homework

Find the area of the given figures. (Note: The diagrams are not drawn to scale).
Round answers to the nearest tenth.

1.



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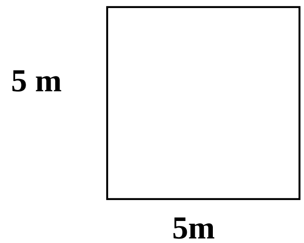

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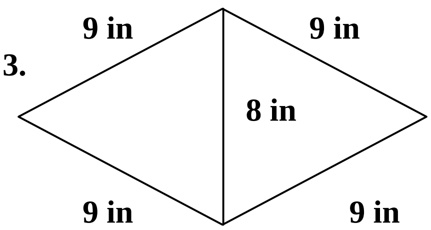

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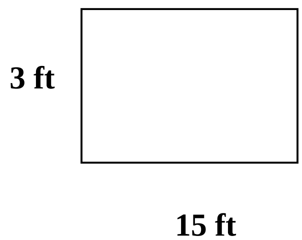
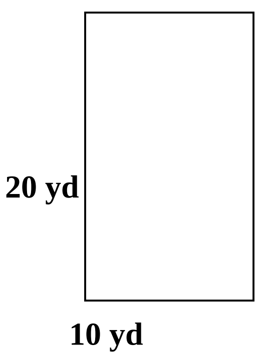
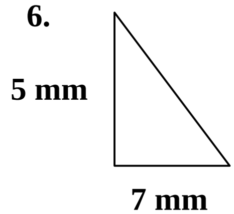


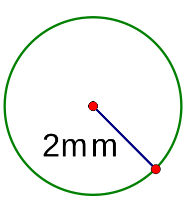
5.



6.



7.



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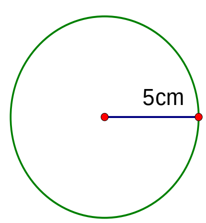
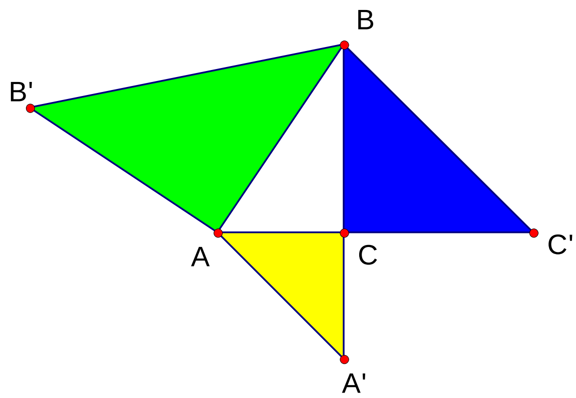
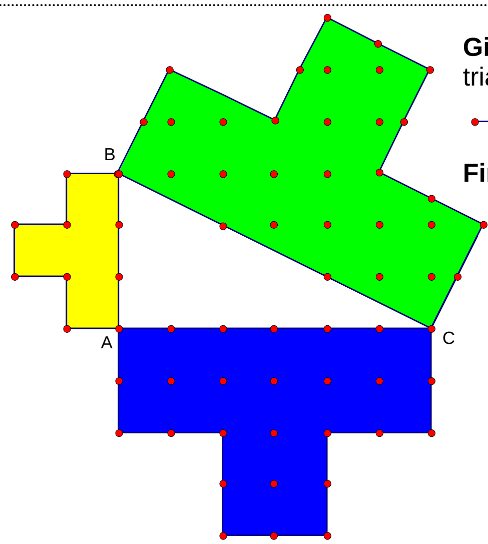


(Resource 7)

Find the Missing Area – Warm Up



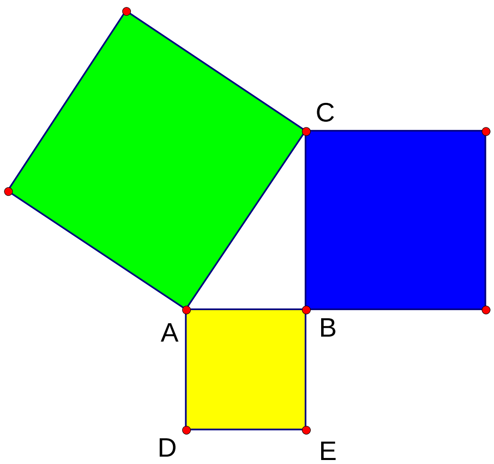
Given: Triangle ABC is right
Triangles AB'B, BC'C and CA'A are similar, right, and isosceles.
Segment AC = 2
Segment BC = 3
Find: The area of Triangle AB'B



Given: The octagons drawn on the triangle are similar.

= 1 unit

Find: The areas of the octagons.



Given: The shapes drawn on triangle ABC are squares.
Segment AB = 20
Segment AC = 22
Find: The area of square ADEB

(Resource 8)

Comparing Areas of Dilated Triangles

fig 1

1. Open Geometer's Sketchpad. Select **Edit** → Preferences. Set the measures to units.
2. Select **Graph** → Show Grid.
3. Select **Graph** → Snap Points.
4. Use the point tool to create a point A and a point B, two units apart on the horizontal axis.
5. Select the points and **Construct** → Segment *(fig. 1).*

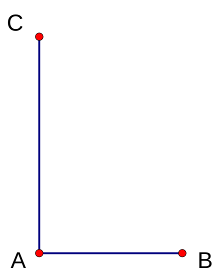


fig 2

6. Select point A and segment AB. Select **Construct** → Perpendicular Line.
7. Use the point tool to construct point C, three units in a vertical direction from point A.
8. Select points A and C and **Construct** → Segment.
9. With no other figures highlighted, select the perpendicular line somewhere outside of segment AC and **Display** → Hide Perpendicular Line *(fig. 2).*

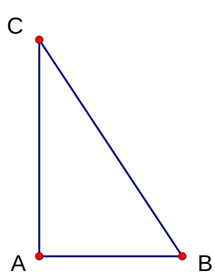
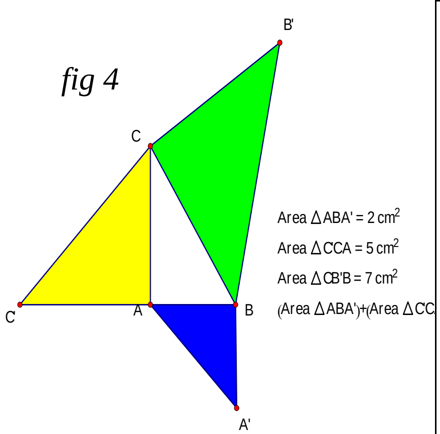


fig 3

10. Construct segment BC, forming a right triangle *(fig 3).*

fig 4



11. Create a 90-degree rotation of segments AC, CB, and BA.
12. Construct segments C'C, B'B, and A'A to create three similar triangles on the legs and the hypotenuse of triangle ABC.
13. Select **Construct** → Triangle Interior for each of the triangles.
14. Use **Measure** → Area to find the areas of triangles C'CA, CB'B, and ABA' *(fig 4).*

**Note: For more information on this, see resource 3, steps 6-10 and 16.*

(Resource 8 Continued)

Selecting vertex B and dragging can now enlarge the figure. The right angle created with the perpendicular line will remain 90 degrees. The calculations will change as the figure is dilated.

The following directions will create tables of you data.

15. Select **Measure** → Calculate. Find the sum of the areas of the two smaller triangles.
16. In order, select the area measurement for the smaller two triangles, the largest triangle, and the sum of the smallest triangles.
17. Select **Graph** →Tabulate. A table appears with the calculations *(fig 5).*

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Area Triangle ABA'	Area Triangle C'CA	Area Triangle CB'B	(Area Triangle ABA')+(Area Triangle C'CA)
2	5	7	7

fig 5

18. As students enlarge the figure, the numbers in the table will change. To "freeze" or fix a set of values in the table, double click on the table.
19. Enlarge the figure and "freeze" some values until desired data is collected *(fig 6).*

Area Triangle ABA'	Area Triangle C'CA	Area Triangle CB'B	(Area Triangle ABA')+(Area Triangle C'CA)
2	5	7	7
4	10	14	14
8	32	40	40
12	50	62	62

fig 6

**Note: Because the Preferences are set to whole units, there may be rounding errors.*

(Resource 9)

Comparing Areas of Dilated Triangles

1. Are the shapes you created on the right triangle similar? Why or why not?
____
____
____
____
____

2. When you change the size of the right triangle, what happens to the shapes drawn on its legs and hypotenuse? Are they still similar?
____
____
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3. Fill in the table below with the information you collected in Sketchpad.

4. Construct a new right triangle with squares drawn on the legs and hypotenuse. Create a table **without** the measurement of one of the smaller triangles. Can you use the information you have to find the missing area?

	Missing Area (Do not calculate on Sketchpad)		

5. Why are squares the most effective shape to use when finding the missing side of a right triangle?
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(Answers for Resource 5)

Pythagoras Wasn't Just a Square

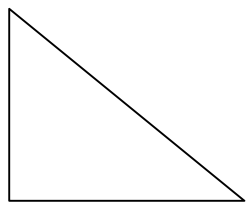
1 ***Sketches will vary.***	Follow your teacher's instructions to form a right triangle with squares on the sides. Sketch your drawing to the left including measurements. How does this relate to the formula for the Pythagorean Theorem? ***Answer should include an explanation that explains how the area of a square is related to its side length.***
2 ***Sketches will vary.***	Sketch the diagram with the three triangles drawn on each side of the right triangle. Do the smaller triangles' areas add up to the same number as the triangle drawn on the hypotenuse? ***Yes.*** Write an equation to show the addition of the areas based on the formula for the area of a triangle: $\frac{1}{2}\boldsymbol{b_1h_1} + \frac{1}{2}\boldsymbol{b_2h_2} = \frac{1}{2}\boldsymbol{b_3h_3}$
3 ***Sketches will vary.***	Draw a quarter-circle or another shape on the sides of a right triangle and sketch your diagram to the left. Calculate the areas of each shape and write your equation: ***Answers will vary.*** Why do you think the Theorem uses squares and the equation $a^2 + b^2 = c^2$ to calculate the missing side of a right triangle? ***Answer should include the observation that squares are the only shape whose side length is always known based on its area.***

(Answers Resource 6)

Homework

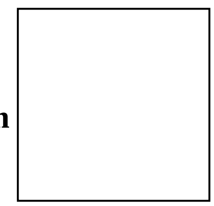
Find the area of the given figures.

1.



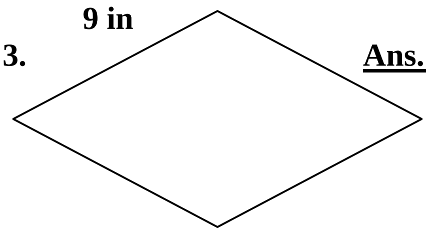
<u>Ans. 40 cm^2</u>

2.

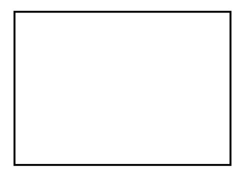


<u>Ans. 25 m^2</u>

3.

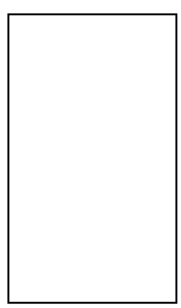


<u>Ans. 64.5 in^2</u>



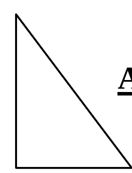
<u>Ans. 45 ft^2</u>

5.



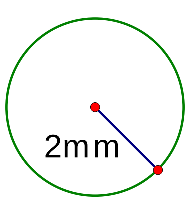
<u>Ans. 200 yds^2</u>

6.

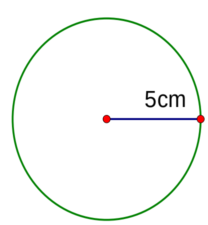


<u>Ans. 17.5 mm^2</u>

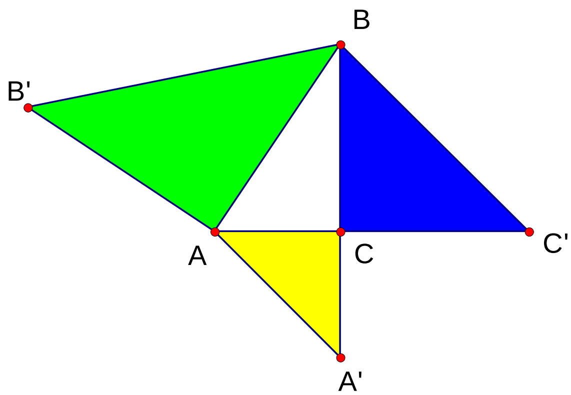
7.
<u>Ans. 12.56 mm^2</u>



8.
<u>Ans. 78.5 cm^2</u>

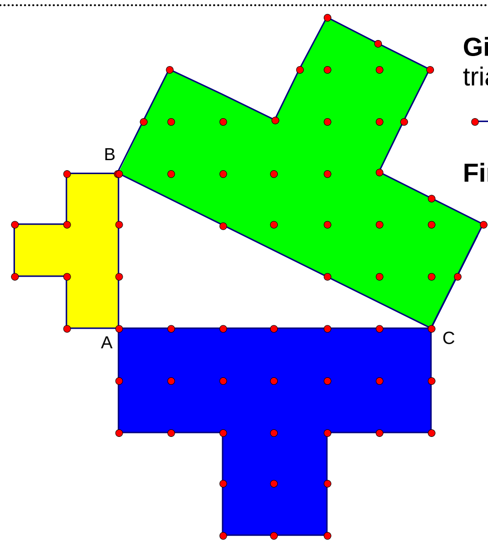


Find the Missing Area – Warm Up



Given: Triangle ABC is right
Triangles AB'B, BC'C and CA'A are similar, right, and isosceles.
Segment AC = 2
Segment BC = 3
Find: The area of Triangle AB'B

Ans: 6.5 units

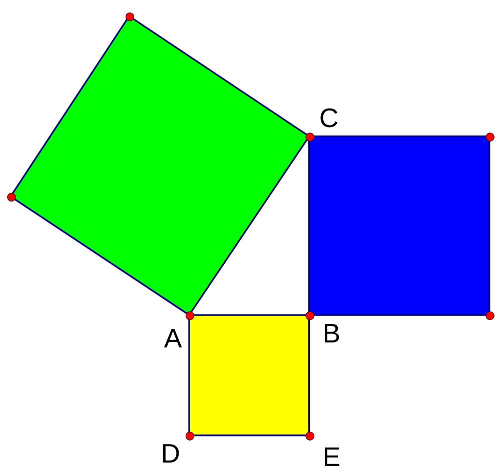


Given: The octagons drawn on the triangle are similar.

= 1 unit

Find: The areas of the octagons.

Ans: 20 units



Given: The shapes drawn on triangle ABC are squares.
Segment AB = 20
Segment AC = 22
Find: The area of square ADEB

Ans: 84 units

(Answers Resource 9)

Comparing Areas of Dilated Triangles

1. Are the shapes you created on the right triangle similar? Why or why not?
Yes. Answer should explain knowledge of similar shapes.

2. When you change the size of the right triangle, what happens to the shapes drawn on its legs and hypotenuse? Are they still similar?
Answer should include that the shapes enlarge, but remain similar.

3. Fill in the table below with the information you collected in Sketchpad.

Table Heading	***Table Heading***	***Table Heading***	***Table Heading***
Answers will vary.			

4. Construct a new right triangle with squares drawn on the legs and hypotenuse. Create a table **without** the measurement of one of the smaller triangles. Can you use the information you have to find the missing area?

Table Heading	Missing Area (Do not calculate on Sketchpad)	***Table Heading***	***Table Heading***
Answers will vary.			

5. Why are squares the most effective shape to use when finding the missing side of a right triangle?
Answer should include analysis of the Pythagorean Theorem.