Title: Understanding Models of Exponential Growth and Decay

Overview:

This unit provides students with a *conceptual* introduction to the exponential function that can preface the traditional treatment that focuses on fluency of algebraic manipulation and translations of the graph.

The Lessons contain a sequenced series of questions that, step by step, lead students to *construct* a conceptual mastery of the exponential function. Specific emphasis is placed upon the defining characteristic of the function. Exponentials model situations where a measured quantity changes by a certain *percentage* each unit interval. This concept is also constructed in terms of a *common ratio*. Students get to discover exponential relationships by using numerical methods, through physical data taking, and by interpreting graphs.

Students will work in cooperative groups of 3-5 with the teacher monitoring each group's progress. The teacher plays the role of a guide or facilitator by asking questions or proposing avenues of exploration to the groups as needed. The Teacher's Reference sheets not only include sample answers, but also key concepts that should be emphasized at that moment. These concepts reappear throughout the series and students should be reminded of the connections. The sheets also address some common student errors and misconceptions. Bold-faced words may be new vocabulary for some students.

The Follow-up Explorations provide additional concept development by exposure to further applications and by analyzing the exponential function in contrast to the linear function. More advanced students can be given these as homework to be discussed later as a class. Otherwise, they can be used as a guided exploration as is done in the lessons. Teachers can assign related drill questions from a text to supplement the lessons if desired.

The Lesson 3 Follow-up Exploration contains material that may not be suitable for all groups. In it, students are exposed to a rationale for the use of the exponential function as a model in physical situations. It is also intended to build readiness for the treatment of exponentials in pre-calculus and calculus.

NCTM Content Standard/National Science Education Standard:

- **Content Standards**
  - **Algebra**
    - Use mathematical models to represent and understand quantitative relationships
    - Identity essential quantitative relationships in a situation and determine the class or classes of functions that might model the relationships
    - Draw reasonable conclusions about a situation being modeled
  - **Data Analysis and Probability**
    - Select and use appropriate statistical methods to analyze data to be displayed on a scatter plot, describe its shape, and determine exponential function using technological tools
  - **Numbers and Operations**
    - Compute fluently and make reasonable estimates
    - Judge the reasonableness of numerical computations and their results
- **Process Standards**
  - **Problem Solving**
    - Build new mathematical knowledge through problem solving
    - Solve problems that arise in mathematics and in other contexts
    - Apply and adapt a variety of appropriate strategies to solve problems
    - Monitor and reflect on the process of mathematical problem solving
  - **Communication**
    - Organize and consolidate their mathematical thinking through communication
    - Communicate their mathematical thinking coherently and clearly to peers, teachers, and others
    - Analyze and evaluate the mathematical thinking and strategies of others
    - Use the language of mathematics to express mathematical ideas precisely
  - **Connections**
    - Recognize and use connections among mathematical ideas
    - Understand how mathematical ideas interconnect and build on one another to produce a coherent whole
    - Recognize and apply mathematics in contexts in real world situations

- **Representation**
  - Create and use representation to organize, record, and communicate mathematical ideas
  - Select, apply, and translate among mathematical representations to solve problems
  - Use representation to mode and interpret physical, social, and mathematical phenomena

Grade/Level:

Advanced Algebra 1, Algebra 2

Duration/Length:

Approximately 3-4 45 Minute class blocks or 2-3 90-minute class blocks

Student Outcomes:

Students will:

- Gain a conceptual understanding of exponential growth and decay.
- Learn the defining characteristic of the exponential function
- Appreciate the numerical behavior of the exponential function.
- Learn to generate exponential models from given initial conditions.
- Learn to generate exponential models from experimentally obtained data.
- Learn to interpret graphs of exponential functions
- Be able to compare and contrast the linear and exponential functions

Materials and Resources:

- TI-83 or TI-84 graphing calculators
- Student worksheets
- 1 or 2 balls of different types for each group
- Measuring sticks or tapes

Development/Procedures:

<u>Lesson 1 Show me the Money</u>

Pre-assessment

Students should be familiar with the concept of a model, linear functions and able to work with percentages and the rule for multiplying powers of the same base. If the students are unable to obtain the correct answers for the sales tax problem at the beginning, the unit is not appropriate.

Launch

Students are introduced to interest bearing accounts, first simple interest that grows linearly, and then compounded interest that grows exponentially.

Teacher Facilitation

The teacher acts as a guide or facilitator as groups of students work on the lessons. It's important to make sure that the students are clear on each step before allowing them to go on to the next. The amount of teacher intervention will vary from group to group. However, there are key points that are mentioned in the Teacher Resource Sheets that should be brought up at the appropriate times. Also, there are some places where direct instruction may be necessary to help students take the next step.

Student Application

Students work in cooperative groups of 3-5. Their work and conclusions should be written on the lesson sheets. Graphs can be made on a calculator, but the students should sketch their results on the lesson sheets. Play money can be used as a manipulative to help students get started.

Embedded Assessment

Students should demonstrate both verbally and with written work that they have understood each step before going on to the next.

Re-teaching/Extension

The Exploration Sheets can be done either for homework or in class in the same way the lesson was conducted.   Students will gain an appreciation for the numerical behavior of the exponential by looking at double and tripling times and by several comparisons between rates and times. The meaning of each part of the model continues to be reinforced. The students will also see how knowledge of exponential behavior can help with decision making. The teacher may decide to assign additional drill problems from a text.

Lesson 2 Value of a Car

Pre-assessment

Students should know the defining characteristic of the exponential function and how to write an equation for exponential growth from a given initial condition and a rate of growth.

Launch

The students learn about the process of depreciation as it relates to the value of a car.

Teacher Facilitation

This lesson is shorter than the first to allow for discussion of the exploration from lesson 1. If time permits, the class can move on to the lesson 2 exploration. The lesson is to be conducted by the teacher as described for lesson 1. The main difference in this lesson from lesson 1 is that the students are asked to interpret and justify the model for exponential decay rather than derive it from data. Students later have the opportunity to examine numerical data in the exploration. Finally, they are asked to make conclusions and generalizations about the exponential function.

Student Application

The students work in groups as was done in lesson 1.

Embedded Assessment

The problem of the car's depreciating value is itself an assessment of the students' ability to apply their understanding of exponential growth to a decay situation. The final questions check to see if the students can recognize and interpret the exponential model using standard constants.

Re-teaching/Extension

The Exploration Sheets can be done either for homework or in class in the same way the lesson was conducted. Students will gain an appreciation for the numerical behavior of the exponential by looking at halving and other fractions. They will also see how knowledge of exponential behavior can help with decision making. Finally, the meaning of each part of the model is reinforced. The teacher may decide to assign additional drill problems from a text.

Lesson 3 Bouncing Balls

Pre-assessment

Students should have basic measuring skills and the ability to work with data on a graphing calculator.

Launch

The students are asked to make some predictions about the behavior of bouncing balls to stimulate their thinking about the experiment.

Teacher Facilitation

The teacher should assist the groups in getting organized and taking measurements. After, the teacher will guide the groups through the analysis and interpretation of the data. The concept of the "common ratio" is emphasized. As was done in lesson 1, students will be deriving the exponential model from data, but this time from a physical process.

Student Application

The groups of students will take height measurements of a ball after increasing number of bounces. Then they will look for a pattern of common ratios, perform a regression, and interpret their model.

Embedded Assessment

The teacher will monitor the groups' progress taking data.

Re-teaching/Extension

The Lesson 3 Follow-up Exploration contains material that may not be suitable for all groups. In it, students are exposed to a rationale for the use of the exponential function as a model in physical situations. It is also intended to build readiness for the treatment of exponentials in pre-calculus and calculus.

Summative Assessment:

The students are asked to find other examples of exponential relationships in the world. A test can also be given that contains "word problems" involving exponentials. These kinds of questions can be found in standard texts.

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Lesson 1: Show me the money. Teacher Resource Sheets.

Spending Money
If you were to buy $100 worth of CD's and the sales tax where you live is 4%, what will be your total cost? What if the sales tax was 8%?

- *Students generally will use the method: Cost = 100 + 100 r where r is the sales tax* ***rate****. Have them consider distributing to get C = 100 (1+r). Emphasize that in the final result, the "1" contributes the original price and that the "r" represents the extra amount added due to the tax.*
- *Also have the students recognize that the number "1+r", in this case 1.04 and 1.08, represents a single multiplier that changes your input number, the price, to the total cost. Emphasize that saying "the cost will be 4%* <u>*more than*</u> *the price" and "the cost is 104%* <u>*of the*</u> *price" are equivalent statements. Point out the subtle difference in the wording.*

Saving Money
When you give your money to an institution like a bank for safekeeping, they will use your money to try to make more money. So, they are willing to pay you to hold your money by giving you **interest** periodically, let's say once a year or **annually**. The interest is generally calculated as a percentage of the money you invest, known as the **principal**. The simplest way to do this is called "**simple interest."** Let's say that the bank will give you 10% simple interest annually and you invest a principal amount of $100.

Make a table of values that shows how much money will be in your account each year for 5 years starting with $100 in year 0. Assume that the bank is paying 10% simple interest paid annually. Hint: The interest is calculated on the principal only.

- *Help the students get started if necessary.*

- *Be sure the students are moving to the simplified form by the end:*

Year	Value
0	100
1	100 + (.1)(100) = 100 + 10 = 110
2	110 + (.1)(100) = 120 + 10 = 120
3	120 + (.1)(100) = 130 + 10 = 130
4	130 + (.1)(100) = 140 + 10 = 140
5	140 + (.1)(100) = 150 + 10 = 150

Describe the pattern of increase. What kind of **relationship** or **function** would best model simple interest? Write the model for the value of the account "V" in terms of the number of years "t".

- *Students should recognize that the account balance is increasing by the same amount each year, the defining characteristic of a linear function. Emphasize this idea of a "defining characteristic" as it will be brought up many times throughout the lessons.*
- $V = 100 + (.1)(100)t = 100 + 10t$

Saving More Money

A more common (and lucrative) method for paying interest is called **"compound interest"**. The value of the account is said to "compound" because instead of basing the interest on the original investment only, each years' interest is based upon the entire value of the account at the time which includes the previous years' interest, as well. So you are earning interest on interest.

Given a principal of $100 and a rate of 10% compounded annually, what will be the value of the account after one year, two years?

- *Help the students get started if necessary.*
- <u>Year</u> <u>Value</u>
  0 100
  1 100 + (100)(.1) = 110
  2 110 + (110)(.1) = 121
- *Most students will probably use the "brute force" method above.*

Continue your calculations to make a table of account values from year 0 to year 5. Can you find any "shortcuts" to your method? Hint" Consider the CD sales tax problem from the beginning of the lesson.

- *Have the students apply the solution of the CD sales tax problem by rewriting the first two solutions as:*
  <u>Year</u> <u>Value</u>
  0 100
  1 100 + (100)(.1) = 100 (1+.1) = 100 (1.1) = 110
  2 110 + (110)(.1) = 110 (1+.1) = 110 (1.1) = 121
  Now use the pattern to get the rest of the values.
  3 121 (1.1) = 133.1
  4 133.1 (1.1) = 146.41
  5 146.41 (1.1) = 161.05

This is an example of ***iteration.*** *Articulate that each year's value is found by multiplying the previous year's value by 1.1, which returns 110% of the value or an increase of 10%.*

How much money is in the account at year 10, 20 and 30? Do you have to calculate all the values in between?

- *The following is likely to require direct instruction to the students. Help the students recognize that the values can be found more efficiently with powers by leading them from more basic iteration method to one based upon powers of 1.1:*

<u>Year</u>	<u>Value</u>
0	100
1	100 x 1.1 = 110
2	110 x 1.1 = (100 x 1.1) x 1.1 = 100 x 1.1^2 = 121
3	121 x 1.1 = (100 x 1.1^2) x 1.1 = 100 x 1.1^3 = 133.1
4	133.1 x 1.1 = (100 x 1.1^3) x 1.1 = 100 x 1.1^4 = 146.41
5	146.41 x 1.1 = (100 x 1.1^4) x 1.1 = 100 x 1.1^5= 161.05

- Continuing the pattern:

10	100 x 1.1^{10} = 259.37
20	100 x 1.1^{20} = 672.75
30	100 x 1.1^{30} = 1744.94
n	100 x 1.1^n

Note that this method does not require the calculation all of the intervening values. Also note the equality of the year number and the power.

Write a model that will give the value of the account for any given year.

- $V = 100\ (1.1)^t$

What is different about this equation compared to others you have seen before?

- *There is a variable in the exponent position.*

Try to write an equation that would give the account values for any year if the principal was \$200 and the interest rate is 10% compounded annually. What if the principal was \$500? Make sure the model works for year 0.

- $V = 200\ (1.1)^t$ $\qquad$ $V = 500\ (1.1)^t$
- *Note that* $1.1^0 = 1$. *This problem explores the meaning of the constant "a" as the initial condition.*

Now write an equation that would give the value of an account starting with a principal of \$100 and an interest rate of 20% compounded annually. For an interest rate of 5%.

- *$V = 100\ (1.2)^t$*
- *$V = 100\ (1.05)^t$*

This problem explores the effect of the base, "b", on the function. This is a good point to reinforce the meaning of the base as (1+r) the "1" returns the original amount and the "r" returns the interest added.

By now, you may have realized that there is a pattern you may follow for this kind of problem:

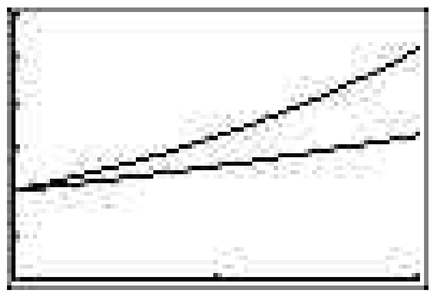
$$V = P\ (1+r)^t$$

where V = account value, P = principal invested and r = interest rate written as a decimal. This kind of equation is called an **exponential equation.** We say that the "value of the account is growing **exponentially**" or "**geometrically**". The defining characteristic of this kind of growth is that the value of the account *increases by a certain percentage* each time period. What role does the "1" play in this equation, in other words, why does it have to be there?

- *Refer back to the tax question at the beginning of the lesson. The 1 returns the original value, while the r only returns the added interest.*
- *Discuss what would happen if you used the model $V = 100\ (r)^t$ in other words, if you forget the 1. This is a common student error in applying the model.*
- *Summarize the lesson by contrasting the defining characteristic of the exponential function with that of the linear function as discussed previously. Be sure to emphasize the difference between growing by the same **amount** each period and growing by the same **percentage** each period.*

Explorations for Lesson 1. Teacher Resource Sheets

1. Consider two different investments. The first starts at $100 and then is compounded annually at an interest of 5%. The second starts at $100 and then is compounded annually at an interest of 10%. Write the equations that model each investment and graph then on your calculator. Start with a window showing year 0 to 10 ($X_{min} = 0$, $X_{max} = 10$). Use the Trace and Table functions to explore the curves.



2. How does the interest rate affect the steepness (slope) of the curve? What information does the steepness at a given point tell you?

- *The larger the rate, the steeper the curve. The steepness tells you how much money is being added to the account in that year.*

3. What is the significance of the y intercept? Why are they the same?

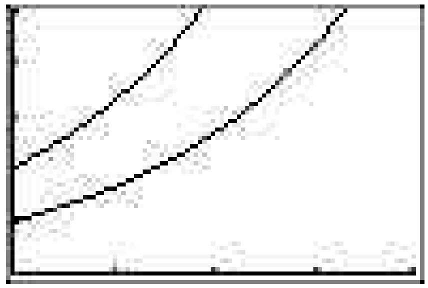
- *The y intercept, which is given by a, describes the initial investment. They are both the same.*

4. Guess how many years it will take for your money to double to $200 and to triple to $300? How long will it take to get to $1000? Use the Trace and Table functions to find out. You will probably have to zoom out.

- *Doubling: About 15 years for 5% and 8 years for 10%. Tripling: About 23 years for 5% and 12 years for 10%*

- *$1000: About 48 years for 5% and 25 years for 10%*

5. Now graph and compare an investment with a principal of $100 and one with a principal of $200. Both are compounded annually at a 5% interest rate. How does changing the principal to $200 affect the graph and values of the accounts? When does each account reach double its original value?



- *The y intercept is shifted up to $200. At every year, the account with a principal of $200 is always worth twice the value of the account with a principal of $100. They both double at 15 years. In fact no matter what the principal, the doubling time will be the same for a given interest rate.*

At what age should you start planning for retirement? Not yet, but this exercise may give you some ideas.

6. Let's say that your goal is to accumulate a million dollars by retirement at age 65 (this is not as much as it may seem.) Assuming that you are going to invest $1000 at age 35, what compounded interest rate (to the nearest whole number percentage) will you need to find to reach your goal? Hint: Try different rates and find your answer by "smart" trial and error.

- *The rate should be between 25 and 26%.*

7. It's unlikely that you will find an investment that will pay as high a rate as you found in question 1. Assuming a more reasonable return, 6%, how long will it take to reach one million?

- *It takes about 119 years*

8. Perhaps you don't have as much time as needed in question 2. How much would you have to invest to start if you want your money to grow to one million dollars in 30 years?

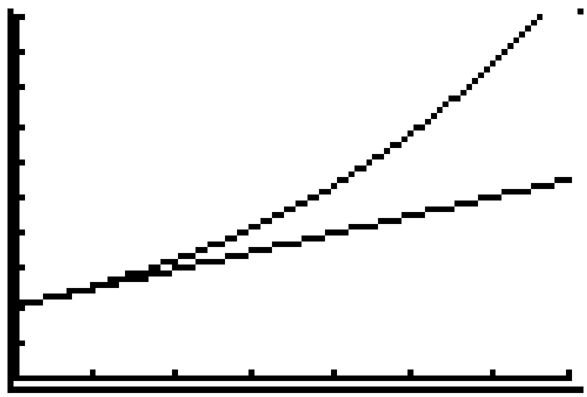
- *About $174,000.*

9. What if you have 40 years to wait?

- *About $97,000*

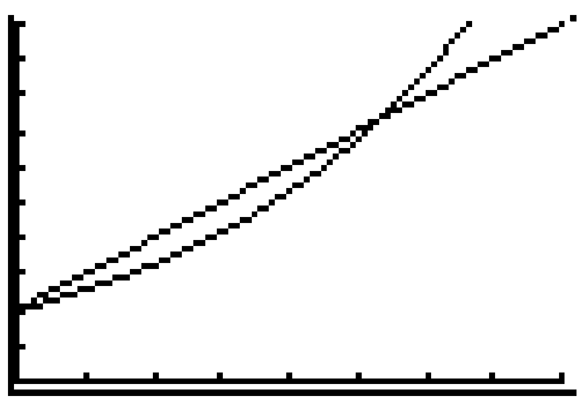
10. At the beginning of the lesson, you learned about simple interest and compound interest. If you have two accounts both starting with the same amount of money and having the same interest rate, will the account that gains compound interest always have more than the account with simple interest? Decide this by graphing the models for a $100 principal and 5% interest for both simple and compound interest.

- *Graph the functions: $y = 100 + 5x$ and $y = 100\,(1.05)^x$. They have the same balance after the first year, but then the compound interest account will always be larger.*



11. How would you decide between two different kinds of accounts that have different interest rates? Try changing the simple interest rate model to reflect 10% interest and keep the compound interest at 5%. Which is better? What should your decision depend upon?

- *Change the first function to: $y = 100 + .1x$ Using trace or the intersect function; students should see that the values of the accounts are equal in about year 26. Their decision could be based upon the length of time they expect to keep the account.*



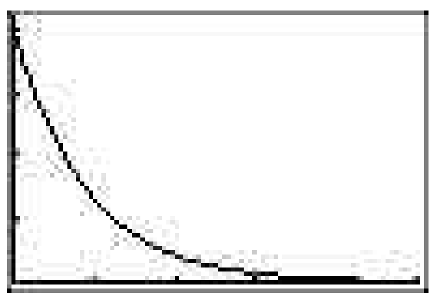
Lesson 2: Value of a Car Teacher Resource Sheets
Most cars lose value each year, a process known as **depreciation.** You may have heard before that a new car loses a large part of its value in the first 2 or 3 years and continues to lose its value, but more gradually, over time. That's because the car does not lose the same **amount** of value each year, it loses approximately the same **percentage** of its value each year. What kind of model would be useful for calculating the value of a car over time?

- *The exponential equation models a situation when a value is changing by a certain percentage each time period.*

If the value of car when new is $20,000 and it depreciates at a rate of 20% each year, explain why the equation that gives the value of the car over time is: $V = 20{,}000\ (.8)^t$.
Hint: There are two ways you might see the above equation:

$$V = 20{,}000\ (1 - .2)^t \qquad \text{or} \qquad V = 20{,}000\ (.8)^t$$

Graph the equation and examine it.



- *Explain that depreciation is represented by a negative rate, in this case -.2.*
- *Emphasize that in the first equation, the rate of change is shown directly whereas in the second, the ".8" represents the percentage of value **remaining**.*

Summary and Generalizing the Exponential Model
The model for something that is increasing or decreasing by a certain percentage every time period is called an exponential equation and has the forms:

$$y = a\ (1 + r)^x \qquad y = a\ (1 - r)^x$$

where a = the initial value, r = the rate of increase or decrease, x = the number of time periods. Or

$$V = a\ b^x$$

Where b = (1 +/- r) and represents the new value's percentage compared to the original. You should be familiar with both forms.

What has to be true about r for the model to be increasing? Decreasing?

- *Increasing: $r>0$, decreasing $r<0$*

What has to be true about b for the model to be increasing? Decreasing?

- *Increasing $r>1$, decreasing $0 < b < 1$*

Explorations for Lesson 2. Teacher Resource Sheets.

1. Consider a car bought at $40,000 that depreciates exponentially, losing value at a rate of 20% per year. Write a model that will give the value of the car each year. Create a table of values for 12 years.

- $y = 40{,}000\ (.80)^x$

X (year)	Y (Value)	Amount of Decrease
0	40,000.00	
1	32,000.00	8,000.00
2	25,600.00	6,400.00
3	20,480.00	5,120.00
4	16,384.00	4,096.00
5	13,107.00	3,277.00
6	10,486.00	2,261.00
7	8,388.60	2,097.40
8	6,710.90	1,677.70
9	5,368.70	1,342.20
10	4,295.00	1,073.7
11	3,436.00	859.00
12	2,748.80	687.20

2. How many years does it take for the value to reach half its value? A quarter of its value? A tenth of its value?

- *Halving takes about 3 years, a quarter the value takes about 6 years and a tenth takes about 10 years.*

3. Calculate the dollar amount of value lost each year? How might all this information influence your decision to buy a new car versus a used one? Do you think that the value of all cars follow an exponential model? Explain.

- *See above for the decreases.*
- *Perhaps buying a used car makes since so much value is lost in the first couple of years. Some cars are collector items and may actually appreciate in value.*

Lesson 3: Bouncing Balls Teacher Resource

What will happen?

Imagine that you drop a ball from 1 meter above the floor. It bounces and reaches a height of .5 meters. What would happen next if the ball lost height according to a linear model? Do you believe that this will occur? What do you believe will actually happen? Explain why.

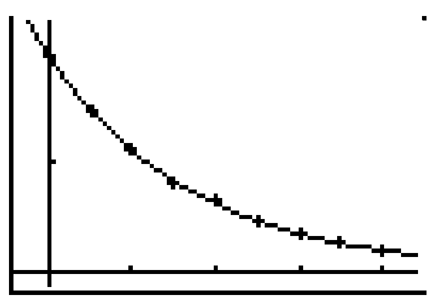
- *If the ball's height followed a linear model, then the ball would hit the floor and stay on the second bounce since it should lose another .5 meters of height. Obviously this would be inconsistent with experience. Many students will guess that it will rise to .25 meters. Be sure that they can articulate the ball lost half of its height on the first bounce so perhaps it will do the same on the second.*

Activity

Work in groups of 2-4 to determine how high a ball will bounce after 1 bounce, 2 bounces, 3 bounces, etc... You will need to set up a scale next to where you will drop the ball. The ball should be dropped from an initial height of 2 meters above the floor. Use the table below to record the number of bounces and the height the ball rises to. Each group should use a different type of ball.

- *Make sure that the floor surface is solid and level. If the students have trouble allowing the ball to bounce several times and stay close enough to the meter sticks for measurement, discuss how you could drop the ball from each previous height to get the next one. Consider having them make at least three drops for each data point to make sure they are getting consistent results. The sample data below is for a golf ball dropped on a hard surface.*

1. Use your calculator to make a scatter plot of your data. What pattern do you see if any? Can you prove your assertion?



- *Students should recognize a descending curve that is similar to the depreciation model. They will prove it as follows.*

2. Now calculate the **ratio** of each height to the previous height and record your results in the table. For example, if the height after the first two bounces were 147 cm and 113 cm, then the ratios would be:

Height after one bounce/Original height = 147/200 = .74
Height after two bounces /Height after one bounce = 113/147 = .77

3. Do you notice any patterns? Remember that real data will always have some uncertainty and/or error involved.

Type of Ball:_____Golf Ball_____________

Number of Bounces	Height (cm)	Ratio
0	200	
		.74
1	147	
		.77
2	113	
		.73
3	82	
		.79
4	65	
		.71
5	46	
		.76
6	35	
		.80
7	28	
		.68
8	19	

- *The students should notice that the ratios are all pretty close to each other. Remind them that this change by a common ratio is the defining characteristic of an exponential relationship. Also explain that this is an alternative way to express the percentage of height (a measure of the ball's energy) retained after each bounce. Relate this to the rate of depreciation in the car examples.*

4. Use the regression feature of your calculator to derive an equation that models your data and graph it over your points. Is the curve a good fit? Discuss the meaning of each part of the equation.

- *Students should use the exponential regression model $y = a\ b^x$. For this data, the regression equation is:*

$$y = 199.5\ (.75)^x$$

- *Discuss how the "a" number represents the initial height, "b" represents the percentage of height that will be retained after each bounce, x is the number of bounces and y is the height after each bounce.*

5. Compare your model with those of other groups. What parts are the same and which are different? Discuss the results.

- *The "a's" should all be close to 200 cm as that represents each group's initial conditions, but b will vary depending upon the "bounciness" of the ball. The number is a measurement of the ball's ability to retain energy while being deformed and then returning to its shape during the bounce. In other words, "b" represents the percentage of height/energy returned after each bounce. The missing energy is converted to heat, which will show as an increase in the temperatures of the ball and the floor.*
- *Have students compare the feel of each ball with its b value. Perhaps they could try to guess which ball goes with which value of b.*

6. Consider the equation, $y = 199.5\ (.75)^x$. Write the equation using the $y = a\ (1 - r)^x$ format. What fact does this form emphasize?

- $y = 199.5\ \ (1 - .25)^x$. *This implies that the ball* ***loses*** *25 % of its height/energy each bounce as opposed to* ***retaining*** *75%.*

Explorations for Lesson 3 Teacher Resource Sheets

1. It is only natural that a compound interest savings account should grow exponentially since the rules of the compounding say to increase the amount of the account by a certain percentage each period, the defining characteristic of an exponential function. But why would a natural phenomenon like a bouncing ball and many others follow this pattern? Examine the graph from the activity. Notice that although the **percentage** of height that is lost each bounce is about the same, the **actual amount** of height lost (as measured in cm) is getting smaller and smaller. The same happened with the depreciating value of the car in Lesson 2. (Think about this!)

   Explain how the changing steepness of the graph supports the above statements.

- *At the start, the curve is steep indicating a rapid decrease. Later, the curve is shallower indicating a more gradual decrease.*
- *Some students will require help making the jump from this step to the next.*

2. The shape of the graph shows that the actual amount of decrease depends upon the **amount remaining** at the time. Many natural and made-made processes share this property. One example is when probability is involved as in flipping a coin. If you drop 1000 pennies on the floor, how many would you expect to come up heads? Of course the answer is about 500. If these pennies are removed and the remaining 500 are dropped, you would expect about 250 to come up heads. Remove those and drop the remaining 250 and so on.

   Explain why the number of pennies removed after each drop depends upon the number remaining. Write a model for the number of pennies remaining after each drop.

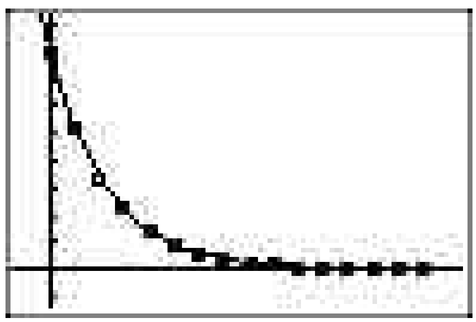
- *The result of heads or tails for a given penny does not depend upon the results of other pennies, a characteristic of many situations that are ruled by probability. Therefore, the fewer pennies there are, the fewer can be removed.*
  $y = 1000\ (.5)^x$. *Discuss the meaning of each part of the model. 1000 represents the number of pennies remaining at time 0, i.e. the initial conditions. The .5 represents both that 50% are retained each throw and that 50% are lost each throw because of the "coincidence" that .5 = (1-.5).*

3. When hot water with a temperature of 100^{o} C is placed into a room temperature environment of temperature 20^{o} C, its temperature will drop until it matches that of the environment. Choosing between a linear or exponential model, which kind of model would you expect the cooling to follow? Explain.

- *Allow students to express their opinions.*

Plot the data below that was obtained by measuring the temperature of water, initially at 100^o C, every minute until it reached the environmental temperature of 20^o C. Describe the rate at which the object is cooling. What kind of model seems to be applicable?

Time (min)	Temp (^{o} C)	Time (min)	Temp (^{o} C)	Time (min)	Temp (^{o} C)	Time (min)	Temp (^{o} C)
0	100	4	34.3	8	22.5	12	20.5
1	72.0	5	29.3	9	21.7	13	20.3
2	53.8	6	26.0	10	21.1	14	20.2
3	42.0	7	23.9	11	20.7	15	20.1



- *The students should choose the exponential model because as the object's temperature gets closer to 20^o C, the rate of cooling becomes more gradual. This describes the appearance of an exponential decay curve.*

Your teacher will show you how to find a regression equation for this situation using your calculator.
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- *Most graphing calculators will not be able to directly find a regression equation for this set of data because the temperature asymptotically approaches the room temperature of 20 degrees rather than 0. In other words, the function is shifted up by 20. However, the standard regression model offered on graphing calculator is $y=ab^x$. To get around this problem, subtract each temperature value by 20 and perform the regression. Finally, add the 20 back in to produce the model $y=ab^x + 20$. At this point, the vertical axis now represents temperature difference rather than temperature.*
- *The regression produces $y = 81.2\ (.65)^x + 20$.*

- *Discuss that the 20 represents the room temperature and why it simply shifts the curve upward. Also note that the "a" rather than being the initial temperature is* *instead approximately the initial temperature **difference** between the water and the room. This implies that the driving force of the cooling is not the actual temperature but the temperature difference that starts at a maximum value and then approaches 0. The "b" tells us that about 35% of the temperature difference is being lost each minute. This will help students understand what follows concerning Newton's Law of Cooling.*

Newton's Law of Cooling tells us that what "drives" the flow of heat from the hot water to the cooler environment is not the temperature of the water, but the **difference between the temperatures** of the water and the environment. At the beginning when the temperature of the water is far from the environmental temperature, the rate of cooling is fast. When the temperature of the water is close to the environmental temperature as at the end, the rate of cooling is slow. Therefore, as with the pennies and the depreciating value of the car, the rate of decrease depends upon how much is remaining. In this case, the difference in temperatures is analogous to the number of pennies remaining or the current value of the car. Whenever the rate of change is related to the amount remaining, there is a good chance that the exponential model will be used.

- *Consider having the students predict what the cooling curve would look like if the cooling proceeded independently of the object's current temperature. This would be true if cooling proceeded linearly.*

- *To take this further, the students can use the Draw Tangent function on the calculator to find the slope of the regression curve at various points. Explain how the slopes represent the rate at which cooling is taking place. If they plot the slopes versus the temperature difference at those points, they will produce a line that implies that the rate of cooling is **directly proportional** to the temperature difference at any given moment. Another extension is to plot the slopes against time. The students will probably be surprised to find that new graph is also an exponential curve as can be verified by a regression. This shows that not only is the temperature difference decaying exponentially, but the rate at which the cooling is taking place is decaying exponentially, as well!*

One Last Question

1. What other situations have you studied or heard about that might be modeled by an exponential equation? Consider growth (getting bigger) and decay (getting smaller). Remember that they will be situations where the **amount of change** each time period will depend upon the amount present at the moment. Also the **percentage change** will remain mostly constant.

- *Radioactive decay, speed of an object coasting to a stop, rates of chemical reactions, population growths, charging and discharging of a capacitor, uncontrolled nuclear reactions, infection rates, an object falling through a fluid approaching terminal velocity etc... Consider performing other experiments like the cooling of water.*

Lesson 1: Show me the money.

Spending Money
If you were to buy $100 worth of CD’s and the sales tax where you live is 4%, what will be your total cost? What if the sales tax was 8%?

Saving Money
When you give your money to an institution like a bank for safekeeping, they will use your money to try to make more money. So, they are willing to pay you to hold your money by giving you **interest** periodically, let’s say once a year or **annually**. The interest is generally calculated as a percentage of the money you invest, known as the **principal**. The simplest way to do this is called “**simple interest.”** Let’s say that the bank will give you 10% simple interest annually and you invest a principal amount of $100.

Make a table of values that shows how much money will be in your account each year for 5 years starting with $100 in year 0. Assume that the bank is paying 10% simple interest paid annually. Hint: The interest is calculated on the principal only.

Describe the pattern of increase. What kind of **relationship** or **function** would best model simple interest? Write the model for the value of the account "V" in terms of the number of years "t".

Saving More Money

A more common (and lucrative) method for paying interest is called **"compound interest"**. The value of the account is said to "compound" because instead of basing the interest on the original investment only, each years' interest is based upon the entire value of the account at the time which includes the previous years' interest, as well. So you are earning interest on interest.

Given a principal of $100 and a rate of 10% compounded annually, what will be the value of the account after one year, two years?

Continue your calculations to make a table of account values from year 0 to year 5. Can you find any "shortcuts" to your method? Hint" Consider the CD sales tax problem from the beginning of the lesson.

How much money is in the account at year 10, 20 and 30? Do you have to calculate all the values in between?

Write a model that will give the value of the account for any given year

What is different about this equation compared to others you have seen before?

Try to write an equation that would give the account values for any year if the principal was $200 and the interest rate is 10% compounded annually. What if the principal was $500? Make sure the model works for year 0.

Now write an equation that would give the value of an account starting with a principal of $100 and an interest rate of 20% compounded annually. For an interest rate of 5%.

By now, you may have realized that there is a pattern you may follow for this kind of problem:

$$V = P\ (1+r)^t$$

where V = account value, P = principal invested and r = interest rate written as a decimal. This kind of equation is called an **exponential equation.** We say that the "value of the account is growing **exponentially**" or "**geometrically**". The defining characteristic of this kind of growth is that the value of the account *increases by a certain percentage* each time period.   What role does the "1" play in this equation, in other words, why does it have to be there?

Explorations for Lesson 1.

1. Consider two different investments. The first starts at $100 and then is compounded annually at an interest of 5%. The second starts at $100 and then is compounded annually at an interest of 10%. Write the equations that model each investment and graph then on your calculator. Start with a window showing year 0 to 10 ($X_{min} = 0$, $X_{max} = 10$). Use the Trace and Table functions to explore the curves.

2. How does the interest rate affect the steepness (slope) of the curve? What information does the steepness at a given point tell you?

3. What is the significance of the y intercept? Why are they the same?

4. Guess how many years it will take for your money to double to $200 and to triple to $300? How long will it take to get to $1000? Use the Trace and Table functions to find out. You will probably have to zoom out.

5. Now graph and compare an investment with a principal of $100 and one with a principal of $200. Both are compounded annually at a 5% interest rate. How does changing the principal to $200 affect the graph and values of the accounts? When does each account reach double its original value?

At what age should you start planning for retirement? Not yet, but this exercise may give you some ideas.

6. Let's say that your goal is to accumulate a million dollars by retirement at age 65 (this is not as much as it may seem.) Assuming that you are going to invest $1000 at age 35, what compounded interest rate (to the nearest whole number percentage) will you need to find to reach your goal? Hint: Try different rates and find your answer by "smart" trial and error.

7. It's unlikely that you will find an investment that will pay as high a rate as you found in question 1. Assuming a more reasonable return, 6%, how long will it take to reach one million?

8. Perhaps you don't have as much time as needed in question 2. How much would you have to invest to start if you want your money to grow to one million dollars in 30 years?

9. What if you have 40 years to wait?

11. At the beginning of the lesson, you learned about simple interest and compound interest. If you have two accounts both starting with the same amount of money and having the same interest rate, will the account that gains compound interest always have more than the account with simple interest? Decide this by graphing the models for a $100 principal and 5% interest for both simple and compound interest.

2. How would you decide between two different kinds of accounts that have different interest rates? Try changing the simple interest rate model to reflect 10% interest and keep the compound interest at 5%. Which is better? What should your decision depend upon?

Lesson 2: Value of a Car

Most cars lose value each year, a process known as **depreciation.** You may have heard before that a new car loses a large part of its value in the first 2 or 3 years and contiues to lose its value, but more gradually, over time. That's because the car does not lose the same **amount** of value each year, it loses approximately the same **percentage** of its value each year. What kind of model would be useful for calculating the value of a car over time?

If the value of car when new is \$20,000 and it depreciates at a rate of 20% each year, explain why the equation that gives the value of the car over time is: $V = 20{,}000\,(.8)^t$.
Hint: There are two ways you might see the above equation:

$$V = 20{,}000\,(1 - .2)^t \qquad \text{or} \qquad V = 20{,}000\,(.8)^t$$

Graph the equation and examine it.

Summary and Generalizing the Exponential Model
The model for something that is increasing or decreasing by a certain percentage every time period is called an exponential equation and has the forms:

$$y = a\,(1 + r)^x \qquad y = a\,(1 - r)^x$$

where a = the initial value, r = the rate of increase or decrease, x = the number of time periods. Or

$$V = a\,b^x$$

Where b = (1 +/- r) and represents the new value's percentage compared to the original. You should be familiar with both forms.

What has to be true about r for the model to be increasing? Decreasing?

What has to be true about b for the model to be increasing? Decreasing?

Explorations for Lesson 2.

1. Consider a car bought at $40,000 that depreciates exponentially, losing value at a rate of 20% per year. Write a model that will give the value of the car each year. Create a table of values for 12 years.

X (year)	Y (Value)	Amount of Decrease
0	40,000.00	
1	32,000.00	8,000.00
2	25,600.00	6,400.00
3	20,480.00	5,120.00
4	16,384.00	4,096.00
5	13,107.00	3,277.00
6	10,486.00	2,261.00
7	8,388.60	2,097.40
8	6,710.90	1,677.70
9	5,368.70	1,342.20
10	4,295.00	1,073.7
11	3,436.00	859.00
12	2,748.80	687.20

2. How many years does it take for the value to reach half its value? A quarter of its value? A tenth of its value?

3. Calculate the dollar amount of value lost each year? How might all this information influence your decision to buy a new car versus a used one? Do you think that the value of all cars follow an exponential model? Explain.

Lesson 3: Bouncing Balls

What will happen?

Imagine that you drop a ball from 1 meter above the floor. It bounces and reaches a height of .5 meters. What would happen next if the ball lost height according to a linear model? Do you believe that this will occur? What do you believe will actually happen? Explain why.

Activity

Work in groups of 2-4 to determine how high a ball will bounce after 1 bounce, 2 bounces, 3 bounces, etc... You will need to set up a scale next to where you will drop the ball. The ball should be dropped from an initial height of 2 meters above the floor. Use the table below to record the number of bounces and the height the ball rises to. Each group should use a different type of ball.

Type of Ball:_________________

Number of Bounces	Height (cm)	Ratio
0	200	
		.
1		
2		
		.
3		
4		
		.
5		
6		
7		
		.
8		

1. Use your calculator to make a scatter plot of your data. What pattern do you see if any? Can you prove your assertion?

2. Now calculate the **ratio** of each height to the previous height and record your results in the table. For example, if the height after the first two bounces were 147 cm and 113 cm, then the ratios would be:

Height after one bounce/Original height = 147/200 = .74
Height after two bounces /Height after one bounce = 113/147 = .77

3. Do you notice any patterns? Remember that real data will always have some uncertainty and/or error involved.

4. Use the regression feature of your calculator to derive an equation that models your data and graph it over your points. Is the curve a good fit? Discuss the meaning of each part of the equation.

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5. Compare your model with those of other groups. What parts are the same and which are different? Discuss the results.

6. Consider the equation, $y = 199.5\ (.75)^x$. Write the equation using the $y = a\ (1 - r)^x$ format. What fact does this form emphasize?

Explorations for Lesson 3

1. It is only natural that a compound interest savings account should grow exponentially since the rules of the compounding say to increase the amount of the account by a certain percentage each period, the defining characteristic of an exponential function. But why would a natural phenomenon like a bouncing ball and many others follow this pattern? Examine the graph from the activity. Notice that although the **percentage** of height that is lost each bounce is about the same, the **actual amount** of height lost (as measured in cm) is getting smaller and smaller. The same happened with the depreciating value of the car in Lesson 2. (Think about this!)

   Explain how the changing steepness of the graph supports the above statements.

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2. The shape of the graph shows that the actual amount of decrease depends upon the **amount remaining** at the time. Many natural and made-made processes share this property. One example is when probability is involved as in flipping a coin. If you drop 1000 pennies on the floor, how many would you expect to come up heads? Of course the answer is about 500. If these pennies are removed and the remaining 500 are dropped, you would expect about 250 to come up heads. Remove those and drop the remaining 250 and so on.

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